QUANTUM MEASUREMENT AND GÖDEL'S PROOF

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Abstract

The measurement problem in quantum mechanics has the character of a fundamental incompleteness within that theory similar to the incompleteness of axiomatic systems in mathematics, discovered and elaborated by Gödel and others. The difficulty of describing the measurement process by the time-dependent Schrödinger equation may reflect the limitations of formal language, and quantum theory may thus require a formalism consisting of two levels of description, one for the dynamics and one for measurement, levels whose relationship resembles that of a calculus and meta-calculus.

A. Introduction

There is an extensive literature on measurement in quantum theory and very substantial disagreement among workers in this field on basic matters of interpretation. The purpose of this paper is to show that the difficulty of including measurement within the quantum formalism has a character similar to the incompleteness which Gödel\(^1,2\) and others demonstrated for certain axiomatic systems in mathematics. That is, the measurement problem may be an instance of a general limitation of formal languages.

To be clear on what is being proposed: It is not here maintained that Gödel's results generate difficulties in physics, because physics necessarily rests upon a mathematics which is incomplete. This kind of argument is so "fundamental" as to be trivial. Nor would an incompleteness theorem necessarily follow from an axiomatization of quantum theory in its present form. Such a result might, however, obtain a discrete formalization of the theory, i.e. to a formalization in which the temporal evolution of the wave-function could be described as the output of a sequential state machine. This would require replacing the continuous time-dependent Schrödinger equation with a recursive algorithm governing sequential changes in a discrete state function. Theories of this type are presently under investigation by Bastin and others\(^3\) but will not here be discussed. What will be offered instead is an informal but extensive parallelism between the physical and mathematical cases, which suggests (but does not prove)


\(^2\) ibid., p. 56.

that the measurement process is self-referential as was Gödel's special formula, and that measurement may be undecidable within the dynamics, occurring only at a meta-level of the formalism.

A general comment, from the mathematical side, for background motivation: It is remarkable, when one considers the enormous effectiveness of mathematics in the natural sciences, that the very substantial body of recent work which deals with the inherent limitations of mathematical systems should fail, thus far, to have any impact on physics and on science in general. The suspicion that this fact is anomalous is widespread, although there have been only a few discussions of the implications of these findings. The result of Komar\(^4\) is the most technical of these; it asserts the undecidability, in quantum field theory, of distinguishing macroscopically between different states of systems with an infinite number of degrees of freedom. This is indirectly connected to the thesis of this paper and will be discussed later. Popper\(^5\) and Lucas\(^6\) have proposed that Gödel's result implies an indeterminism in classical physics. This indeterminism is not, however, found within the physical theory itself, but rather is suggested "from the outside" by computational argument. The proposal of this paper, by contrast, is that an indeterminism which resembles Gödel's results, while not recognized as such, is actually already imbedded in the theory, specifically in the measurement problem of quantum mechanics. Schlegel\(^7\) has linked the incompleteness of logical systems with the uncertainty relations of quantum theory, attributing both to the consequences of self-reference. This suggestion differs from, but is close in spirit to, the present proposal; the "measurement problem" is related to but nonetheless distinct from the interpretation of the uncertainty equations. Pattee\(^8\) has also drawn some connections between logical incompleteness and quantum uncertainty, and other observations of his, on the relations between the dynamics and the control of systems, as will be shown later, are of special relevance to the argument of this paper.

This is some of the background for the proposal we shall develop. In the next section, the measurement problem is defined and then interpreted in terms of mathematical undecidability.

B. The Measurement Problem

We consider the first systematic treatment of the measurement problem, that of Von Neumann\(^9\), which speaks of two fundamentally different processes by which a system can change:

P1. If the observed system, \(O\), in the state

\[\psi (x) = \sum \psi_k u_k (x)\]  \hspace{1cm} (1)

is subjected to a measurement of \(A\), whose eigen-functions are the set of \(u_k\), then a transition ("projection") \(\psi \rightarrow u_n\) occurs, for some \(n\), with probability \(|\psi_n|^2\).

P2. If the system is undisturbed during a time interval, then in that interval, its state changes according to a dynamic law: the time-dependent Schrödinger equation (TDSE).
\[ \text{i} \hbar \left( \frac{\partial \psi}{\partial t} \right) = H\psi \] (2)

The "measurement problem" consists of the impossibility of describing P1 in terms of P2. If one considers the measuring system, M, with wave function \( v(y) \), explicitly in this treatment, one has initially,

\[ \psi(x,y) = v_0(y) \sum_k \psi_k u_k(x) \] (3)

where \( v_0 \) is the prepared state of M, and finally, via P2, since \( v_0 u_n \rightarrow v_n u_n \)

\[ \psi(x,y) = \sum_k \psi_k u_k(x)v_k(y) \] (4)

P2 thus does not leave M in any definite eigen state. By contrast, however, P1 reduces the superposition, giving \( u_n(x)v_n(y) \) for some \( n \) with probability \( |\psi_n|^2 \).

Since the classical work of Von Neumann, many alternative formulations of quantum measurement have been offered. These have included hidden variable theories, proposals of non-linear modifications of the TDSE, the assertion of the indistinguishability of superpositions and measurement-produced mixtures, the postulation of intervention by consciousness or of the necessity of retaining a classical description of measurement, the statistical interpretation of the theory as applying to ensembles and not individual systems, etc. No one of these has met with general acceptance, and since the projection postulate is retained by some recent investigators\(^{(10)}\) (despite the more common view that measurement produces a mixture), we shall make use of the Von Neumann account, for its convenience. Nonetheless, the present argument is not critically dependent on projection and relates also to several other interpretations of quantum measurement. It cannot, in any case, be given a rigorous statement in the context of a continuous theory, and so the differences between the formulation of Von Neumann and more contemporary approaches are, for the present, of secondary importance.

To set out the parallelism between quantum measurement and Gödel’s proof: Given the initial superposition, the measurement event (P1) is not covered by the formalism of the quantum mechanical differential equation (P2), or more precisely, the observation or M in a definite eigen state, \( v_n \), or the composite system, \( M+O \), in \( u_n v_n \), is not derivable via the TSDE, just as the formula Gödel constructed cannot be proven within the axiomatic system of the arithmetic calculus. That is, a wave function corresponds to a well-formed formula (wf) in the calculus. The temporal evolution of the wave function, resulting in some final state, may consequently be modelled by the sequence of wf in a proof of a theorem, i.e. the differential equation, P2, which generates a sequence of \( \psi \)'s, acts like the rules of inference in the axiomatic system, which allow the generation of sequences of well-formed formulae. The particular route taken in the proof corresponds to the detailed construction of the measuring apparatus with its specific Hamiltonian. A wf which is decidable is one which can be reached by a proof sequence from the axioms and/or previously proven theorems in the calculus, and thus corresponds to a \( \psi \) predictable via P2 from some initial wave
function. Gödel’s results make use of (and clarify) the distinction between decidability and truth, and we may make a parallel distinction in our physical theory, between predictability (via P2) and the possibility or fact of occurrence. Gödel constructed a wf which was undecidable, but, by meta-arithmetic argument, true; analogously, we propose that measurement (P1) occurs but is not, in the sense of P2, decidable.

In such a view, our physical theory would, of necessity, have at least two levels, and P1 would not be on the “base” level which is described by the dynamical law P2, just as we distinguish between calculus and meta-calculus. A similar distinction, although without reference to Gödel’s result, has been proposed by Weyl(11):

The ‘physical process’ undisturbed by observation is represented by a mathematical formalism without intuitive interpretation; only the concrete experiment . . . can be described in intuitive terms. This contrast of physical process and measurement has its analogue in the contrast of formalism and meaningful thinking in Hilbert’s system of mathematics. As it is possible to formalize an intuitive mathematical argument, so is it true that measurement . . . may be interpreted as a physical process. In doing so one has to extend the original system . . . But as soon as we want to learn something about (the extended system) that can be told in concrete terms, then the undisturbed course of events as ruled by the dynamical law must again be disrupted . . .

Measurement is thus defined at the level of P1, but, on the level of P2, does not occur. This accords with the common notion that there must be some sense in which P2 is sufficient and there are no “measuring instruments” or “observers” at all. While P1 would not be describable by P2, the two processes also would not conflict, just as Gödel’s formula cannot lead to results which contradict theorems in the calculus. (Otherwise, the Gödelian formula would be disproved.) We thus have a means of explaining how the results of P1 and P2 can be distinct, and yet not generate a contradiction: P1 is taken as a meta-level statement to a (P2) formalism which is inherently incomplete.

This is a very general statement of the present proposal. It would, though, be desirable to recast our description of quantum measurement in a form which clearly exhibits a self-referential character, and, moreover, one which implicitly asserts its own undecidability. As stated earlier, this would require replacing the wave functions and differential equation of P2 with discrete state representations and recursive operations on them, i.e., we would require a theory more closely resembling the mathematical systems to which Gödel’s results are applicable. Such a theory is not, however, essential to an informal exposition of the present thesis. It is commonly accepted that results from the theory of computation have general implications for physics, even though Turing machines and the like are obviously mathematical, and not physical, constructs, and operate in a discrete, rather than continuous mode. (Popper, for example, takes computational arguments and Gödel’s proof as relevant to classical physics.)

Nonetheless, it will be helpful to indicate roughly how Gödel’s result might be relevant to the measurement problem in a discrete theory(12). The formula
Gödel constructed can be represented by the following statement:

$$y: (x) \sim \text{Dem} (x, 'y')$$

(5)

where "y:" indicates that y is the Gödel number of the formula which follows, "Dem (α, 'β')" signifies that a string of formulae whose Gödel number is α is a proof of the formula given by a function whose value is β, and where the statement says that for all x, x does not stand in relation Dem to y. (This, of course, is an extreme simplification of Gödel's construction. For example, the second argument of Dem is really a complex function designed to ensure that the Gödel number of the proposition is the same as the value of the function.)

We can now assume a quantum theory in the form of a predicate calculus in which it is possible to assign a unique Gödel number to each wave function. This is not unreasonable, since, for example, in quantum field theory, a state is named by a sequence of integers representing the occupation numbers for all the degrees of freedom in the system.(1) Moreover, just as one can, in principle, specify a state by a recursive procedure which describes its physical preparation, one can represent the dynamical law P2 as a recursive procedure, analogous to Dem in (5). This procedure generates sequential changes in the integer strings representing wave functions, and may be derived from the Hamiltonian describing the interaction of the measuring and measured systems. We should stress that there is no intention here of ascribing any physical significance to Gödel numbers; they are "artifacts" of the particular means we use to map states and transformation rules onto integers and relations between integers.

Now, consider a formula of type (5), with P2 replacing Dem:

$$y: (x) \sim \text{P}2 (x, 'y')$$

(6)

While expressing, at the level of the calculus, a purely arithmetic proposition, the meta-level significance of this formula would be the following: There is no sequence of wave functions, x, which stands in relation P2 to wave function y. Moreover, this assertion is implicit in y itself. The possibility of mapping wave functions onto the integers and the translating of P2 into a recursive procedure can thus give rise to the existence of "unrealizable" (via P2) wave functions. Writing (6) more explicitly, we have:

$$u_n : (N) (ψ^1 ψ^2 \cdots ψ^N) \sim \text{P}2 (ψ^1 ψ^2 \cdots ψ^N, 'u_n')$$

(7)

i.e., there is no wave function sequence, $ψ^1 ψ^2 \cdots ψ^N$, where $ψ^i = ψ(t_i)$, beginning with the linear superposition $ψ^1 = \sum ψ_k u_k(x)$, which results, via P2 (with a Hamiltonian of eigen functions, $u_k$), in the pure state $u_n^{(1)4}$. (To include both measuring and measured systems, replace $u$ by $v$ in the above, with appropriate subscripts.) Thus, in a discrete quantum theory, if a formula of form (6), with interpretation (7), can legitimately be constructed, we would have a formula representing a wave function, $u_n$, which is true but undecidable. This would be interpreted as a wave function which can occur but not be predicted via P2.

We suggest that such a formula, (7), could represent the process of measurement: Its Gödel number denotes a wave function, and it simultaneously declares, at a meta-level to the formalism, that this wave function cannot be arrived at from the initial state by the standard recursive operation used for proof in the calculus. We can regard the process P1 as equivalent to the adoption of this Gödelian formula as a new axiom in the calculus.(15) Subsequent theorems, or wave functions, in the calculus could then evolve from this axiom via P2.
That measurement cannot be encompassed by the laws of dynamics is also an implication of Popper's claim for indeterminism in classical physics. Popper provides three arguments, one directly from Gödel's result, the others of similar nature, for the assertion that a machine cannot have knowledge of its own state before the state has passed. In terms of our previous discussions of measurement, this means the impossibility of M being in state \( v_n \) and simultaneously exhibiting this fact externally. Yet this is required by P1. This paradox can be resolved in a two-levelled theory, where P1 is represented in some form similar to (7). This insures that the measurement result is not derivable from the dynamics, and simultaneously, that it nonetheless "happens". Thus, in a quantum theory, with its two modes of describing changes of state, we can resolve Popper's paradoxes while in a "single-levelled" classical theory, we cannot. Or, to put it differently, the duality of the quantum description of state changes is actually a solution to a logical flaw in classical theory. This suggestion, as we shall show later, is supported by other essential differences between the two theories.

To summarize: In a calculus representing quantum theory, where wave functions are mapped uniquely onto integers, and where the time-evolution of the wave function (P2) is represented by a recursively defined predicate like Gödel's "Dem", there may exist some Gödelian formula whose physical interpretation suggests the results of measurement\(^{16}\). That is, a calculus representing the operation of P2 can give rise to an undecidable (but nonetheless true) formula which can be interpreted as P1.

More, however, is required. We took, as the outcome of the measurement, one of the eigen states, \( u_n \). There are, in general many \( u_n \)'s which can be the outcome of the measurement, and for each such eigen state, we must be able to construct a formula of type (7). It should be possible to adopt as a new axiom, any (and only) one of this set of formulae, and to do so with probability \( |\psi_n|^2 \). Clearly we require here a predicate calculus not of a conventional type, such as was the subject of Gödel's proof, but one which embodies what Putnam and Bub have referred to as "the non-Boolean logical character" of quantum mechanics \(^{17,10}\). We do not have a Gödelian result for this kind of calculus, and so we have given only a general indication of how measurement might be shown to be undecidable. We have been led naturally, in the present discussion, to assume projection. This is not surprising since, as Bub observes\(^4\), "the projection postulate is the appropriate rule for the non-Boolean logical space of quantum mechanics."

So far, we have suggested how, in a non-standard calculus, we might obtain the undecidability of the measurement process. This argument can be developed further by examining (a) other aspects of the present quantum theory, particularly the irreversibility of measurement, and (b) other interpretations of the measurement problem.

C. The Irreversibility of Measurement; Relationship to Other Views of Quantum Measurement

A modification of the calculus involving the addition of a Gödelian formula as a new axiom will leave the calculus still incomplete. It will have its own undecidable formula (at least one). We can, therefore, envision a sequence of calculi,
each bigger than the previous one by the assumption of the latter's Gödelian formula and all theorems which follow from it. Such a sequence of calculi parallels a sequence of measurements in that it calls to mind a particular feature of quantum measurement discussed by Von Neumann but also retained in most later formulations: the fact that, while all processes described by the time-dependent Schrödinger equation are reversible, the changes produced by P1 are irreversible.

The unique time direction for P1 events can therefore be modelled by the direction of growth of the calculus, modified by successive Gödel formula additions. If we restricted ourselves to P2 processes, we do not obtain this irreversible expansion, but rather stay within the bounds of our original calculus. (Reversibility requires only that there exist a Turing machine capable of performing the inverse computational sequence.)

Pattee(8) has offered a suggestion very close to the present one:

"Why are there two levels of structure and description necessary for any prediction and control process? The basic reason is that in order to predict how a system will behave we must assume it can behave only one way according to its dynamical law, without the possibility of some alternative behaviour. On the other hand, in order to speak of controlling a system we must assume that alternative behaviours are possible. How can a system have control alternatives when no dynamical alternatives exist? This is the same conceptual problem that has troubled physicists for so long with respect to irreversibility. How can a dynamical system governed deterministically by time-symmetric equations of motion exhibit irreversible behaviour? . . . These questions appear paradoxical only in the context of single-level descriptions. If we assume one dynamical law of motion that is time reversible, then there is no way that elaborating more and more complex systems will produce irreversibility under this single dynamical description. I strongly suspect that this simple fact is at the root of the measurement problem in quantum theory, in which the reversible dynamical laws cannot be used to describe the measurement process. If the event itself is time-symmetric, the record of the event cannot be, for it is primarily by records that we give time a direction. This argument is also very closely related to the logician's argument that any description of the truth of a symbolic statement must be in a richer metalanguage (i.e. more alternatives) than the language in which the proposition itself is stated."

One can replace the word "controlling" in the third sentence with "measuring". Measurement also intrinsically assumes the possibilities of alternative results, and not predictable dynamical behaviour, else why, if one can write down the initial state, would one perform such an operation? Now if the condition for measurement is independence from the dynamics, how is this actually to be accomplished without contradiction even within a two-levelled theory? Here is the essential reason why the measurement event must intrinsically proclaim its dynamical undecidability, thereby simultaneously securing its truth at the higher level.

Pattee's speculations are supported in part by developments in the theory
of computational complexity. Chaitin\textsuperscript{(18)} has shown that in a formal system, where theorems may be represented as integer strings, it is impossible to demonstrate that any string has a complexity greater than that of the system's axioms. (Here complexity is a technical term indicating the size of the minimal program required to generate the string and thus its effective information content.) For our calculus which models a dynamical system, this states essentially that entropy will not increase beyond its initial value\textsuperscript{(19)}, and that measurement, since it is associated with entropy increase, cannot be described by the system dynamics. Chaitin's finding is proven via a Gödelian argument and actually represents an implication of Gödel's proof for information theory. It is likely that this only touches upon the deep connections which may exist between information theory, mathematical undecidability, and the measurement problem.

We have suggested that measurement, if irreversible, cannot be encompassed within a dynamical description, or its formal mathematical equivalent. That irreversibility must actually be property of measurement is also required on information-theoretic grounds from a different point of view. This, in essence, was the solution to the problem of the Maxwell demon\textsuperscript{(20)}. If measurement did not increase entropy, the demon would be able to defy the second law of thermodynamics by collecting only rapidly moving particles in one of the two chambers. Moreover, while this argument seems to be a classical one, this is not the case. If we stay within the realm of classical dynamics, the particle can, in principle, be observed with an arbitrarily small exchange of energy and thus finite information can be gained at the cost of an infinitesimal increase in entropy. The use of thermodynamic arguments, classical in themselves, begs the question, since the entropy increase of the Second Law cannot be derived for Newtonian mechanics without introducing an "extraneous" statistical postulate\textsuperscript{(21)}. The dynamics cannot give rise to irreversibility, and thus any gain in information. By contrast, irreversibility in quantum theory is inherent in the formalism — in the description of measurement.

Classical mechanics thus can be said to have a measurement problem itself, but one not formally recognized. To avoid infinite regress in the description of measurement, and paradoxes either of self-reference or involving the issue of reversibility, one must assume that at some point, the perturbation by the measurement can be ignored, or introduce a statistical postulate as an arbitrary addition to the dynamics. If neither is allowed, one can simply accept the need for the two-levelled theory. Quantum theory "owns up", as it were, to this situation and thus its measurement problem is prominently displayed.

Bohr's position\textsuperscript{(22)} that the measurement instrument requires description in classical language and interpretations of quantum theory which posit a special role for consciousness (e.g. Wigner\textsuperscript{(23)}) do essentially just this: they introduce a second level into the theory. The classical instrument or the role of consciousness is, in effect, taken as a meta-level with respect to the dynamics since, after all, the instrument can be described by the TDSE and consciousness is associated with a material base for which P2 holds rigorously. Thus, the suggestion of a significant connection between Gödelian undecidability and the measurement problem makes the same distinctions as these proposals, but in a more formal
way. It avoids both the arbitrariness of a dual classical/quantum description of measurement (or a split of uncertain location between the two), and the extreme anthropomorphic character of arguments invoking the consciousness of observers. We can relate our thesis also to the proposal of Everett(24). The adoption of one of several Gödelian formulae allows our original calculus to “split” into multiple new calculi, corresponding to all the possible states $u_n$. This is more conservative than multiplying whole universes.

D. Conclusion

It is remarkable that all the interpretation of measurement in quantum theory is still, at this late date, a subject of much controversy; therefore, perhaps it is justified to explore radically new approaches to this problem, even if they can only be set out in a tentative manner. In the absence of a discrete model for quantum theory and an undecidability proof for a non-Boolean predicate calculus, the parallels drawn in this paper must inevitably be informal and incomplete. Still, in a general way, the measurement problem has many features suggesting an incompleteness arising from self-reference, and this is unlikely to be coincidental. At the very least, the lack of serious consideration of this possibility in the measurement problem literature is puzzling. This paper is offered in the hope that it might stimulate such discussion, and as a preliminary statement of a possible explanation of the quantum measurement problem in terms of the intrinsic limitations of formal languages.

References

12. We follow the style of Gödel's proof as presented by Nagel and Newman, but wish to point out that a relationship between mathematical undecidability and quantum measurement does not depend upon the possibility of a strict transposition of Gödel's constructions into the language of physics. There are ways of expressing the incompleteness of logical systems which are more general than the particular approach used by Gödel. This approach is employed only to illustrate how the present proposal might be expressed more concretely.

13. The conversion of such a sequence, if finite, to a single Gödel number, is trivial. For systems with an infinite number of degrees of freedom, one can, as Komar\(^4\) notes, name a state by specifying a recursive procedure which allows one to generate the \(n+1\)^{st} term in the integer sequence from the first \(n\) terms. Since such a procedure would itself be expressed in terms of the symbols of the calculus, it too could be designated by a unique Gödel number, as can also any possible sequence of wave functions.

14. The same operator would have to take \(u_n\) into itself, so if, in addition, it took the linear combination into \(u_n\), the time-reversibility of the TDSE would lead to the anomaly that starting from \(u_n\) one could, with a single Hamiltonian operator, arrive either at \(u_n\) or an arbitrary linear combination of eigen states.

15. This would be in disagreement with "approximate solutions" which assert the indistinguishability of the results of P1 and P2, since (in our hypothetical discrete quantum calculus) a wave function derivable via P2 from the initial state would have a Gödel number distinctly different from that corresponding to \(u_n\).

16. As an aside, we wish to mention the claim of Komar that "there is, in general, no effective procedure for determining whether or not two arbitrarily given states of a quantum system having an infinite number of degrees of freedom are macroscopically distinguishable". This may have direct bearing on the measurement problem, as follows: Assume that one can represent the conversion of the initial superposition into a mixture via P1 by a recursive procedure, just as we have done for P2. Komar's result implies that there would be no way to decide whether the macroscopic consequences of P1 and P2 are the same (and, as a corollary, also, no way to establish the microscopic identity of the two results). This would seem to settle the arguments surrounding the "approximatist position" in a most interesting way: Neither those who claim that P1 and P2 are experimentally indistinguishable, nor those who deny this claim, are correct; the matter is undecidable. (We must, of course, assume that both P1 and P2 apply to individual systems and not ensembles.) This argument is different from that presented in this paper, since Komar's result depends critically on the presence of an infinite number of degrees of freedom, but the measurement problem is not so restricted.


19. A possible source of confusion here is that information, complexity, and randomness (hence entropy) all have the same sense in Chaitin's terminology (which is the same as that originally used by Shannon, as discussed by Brillouin), while it is perhaps more familiar to regard information as negative-entropy. One can take either point of view: a random process is one about which an observer has minimal information, but a string of numbers produced by such a process requires a maximal amount of information for its specification.

21. Brillouin, in his own exorcism of the demon, claims that his result, being independent of Planck’s constant, “can be given along classical lines without the introduction of quantum conditions”. Yet, his defining equation for entropy, $S = k \ln P$, depends critically on a quantum assumption to justify the finite number (P) of discrete states. Also the extraneous statistical postulate is evident in Brillouin’s discussions of Szilard’s paper: He adds up the information gained from finding molecules in V1 and in V2, and notes, “Each individual operation may yield information smaller than or larger than the entropy increase per operation”. In quantum theory, the possibility of finding the molecule in V1 and V2 is built into the wave function description, and hence the statistical assumption isn’t an extraneous one, while in a classical dynamical description, a single measurement by the demon can violate the Second Law. That the Second Law “can be understood and satisfactorily interpreted only on the basis of quantum mechanics” is argued by D. Gabor, in “Light and Information”, Progr. Opt., 1, 109-153 (1964), who also cites the similar and earlier view of Born.

