A TALE OF TWO THEOREMS

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Abstract

In combination the Okishio theorem (1961) and a theorem due to John Roemer (1981) together imply that a capital-saving technical change could simultaneously (1) reduce production costs and therefore be adopted by profit maximizing capitalists, (2) make the economy less productive, yet (3) raise the rate of profit even while the real wage remains constant. But how can a technical change which makes the economy less productive make capitalists better off if workers are no worse off? This article resolves this conundrum, and provides a useful procedure to calculate how much any technical change increases the productivity of the economy as a whole.

The Okishio theorem (1961) is well known. It says that if we hold the real wage constant any “viable” technical change, i.e. any technical change that reduces production costs at current prices, must necessarily raise the equilibrium rate of profit in the economy. John Roemer proved a theorem (1981) that is less well known. It says that when the rate of profit is positive there can be capital-saving, labor-using technical changes that are viable but also “retrogressive,” i.e. make the economy less productive. The model and assumptions Okishio and Roemer both use are identical, well known, and essentially the same as those used by Marx. Formally: Goods are produced by homogeneous labor and circulating capital defined by a matrix of produced inputs, $A$, and a vector of direct labor input coefficients, $L$. The real wage is expressed as a column vector, $b$, of goods per unit of labor. $A$ is assumed to be non-negative, indecomposable, and productive, i.e. satisfying the Hawkins-Simon condition. In which case, how can both theorems be true? How can a technical change which makes the economy less productive raise the rate of profit if the real wage remains the same? If the economy is less productive isn’t the economic “pie” smaller? In which case, if workers do no worse, how can capitalists do better?
1. Labor Values as Social Costs

When there are no other scarce primary resources except homogeneous labor, the only input society has any reason to “economize” on the use of is labor, and the vector of labor values, \( \mathbf{V} = \mathbf{L}[\mathbf{I}-\mathbf{A}]^{-1} \), serves admirably to indicate whether this has, or has not occurred. Since the labor value of good \( j \), \( v(j) \), includes not only the amount of labor needed directly to make a unit of \( j \), but the amount needed indirectly to make the non-labor inputs needed to produce a unit of \( j \) as well, \( v(j) \) tells us the total amount of labor required to make a unit of good \( j \). Suppose a technological change meant that the labor value of some good fell, and the labor value of no other good rose. That would mean that it requires less of our labor time to produce a unit of one good, and no more of our time to produce a unit of any other good. As long as we consider leisure preferable to work, and as long as there are no other scarce resources society should economize on, this would be a socially beneficial change in the conditions of production, which Roemer calls a *progressive* change. If, on the other hand, a change resulted in the labor value of some good rising, and the labor value of no good fell, that would be counterproductive, or what Roemer calls *retrogressive*, from a social perspective. If a technological change left the vector of labor values unchanged, clearly the change would be *neutral* from the point of view of social efficiency. Formally: A technical change is progressive iff: \( \mathbf{V}' \leq \mathbf{V} \). A technical change is retrogressive iff: \( \mathbf{V}' \geq \mathbf{V} \). And a technical change is neutral iff: \( \mathbf{V}' = \mathbf{V} \).

Moreover any technical change must necessarily be one of the three because it is impossible for a single technical change to lower the value of one good and raise the value of another good. To see this, suppose a technical change in industry \( j \) lowers \( v(j) \). If the economy is indecomposable good \( j \) must enter into the production of every other good, either directly or indirectly. Since the input coefficients have not changed in other industries where no technical changes have taken place, the value of every other good must fall because the value of good \( j \) used to make every other good is lower. Similarly, if the technical change raises \( v(j) \) then the value of every other good, \( v(i) \) \( i = 1,2\ldots n \) must also rise.

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1 Here homogeneous means not only is every hour of labor equally productive it is also equally undesirable to perform.
2. Roemer’s Theorem and Adam Smith’s Second Invisible Hand

If $r = 0$ there is a one to one correspondence between viable and progressive technical changes. In which case Adam Smith’s second invisible hand works perfectly: Profit maximizing capitalists can be relied on to adopt or reject new technologies for us because when they serve their private interest they serve the social interest as well.\(^2\)

*Roemer’s Theorem:* However, if $r > 0$:

(i) All CU-LS technical changes that are viable are progressive, but there are CU-LS progressive changes that are not viable.

(ii) All CS-LU technical changes that are progressive are viable, but there are retrogressive CS-LU changes that are viable as well.

(iii) As the rate of profit rises more and more CU-LS progressive changes become non-viable, and more and more CS-LU retrogressive changes become viable.\(^3\)

In other words, in the case of CU-LS changes, while capitalists can be relied on to introduce some progressive changes, they will not necessarily introduce all CU-LS progressive changes. And in the case of CS-LU changes, while capitalists will introduce all progressive changes of this variety, they may also introduce retrogressive CS-LU changes as well.

When and why will capitalists fail to carry out the social interest regarding choice of technology? If the rate of profit is zero, there is a one-to-one correspondence between viable and progressive changes, and Adam Smith's second invisible hand reigns supreme. Why? If $r = 0$ the relative price vector,

\(^2\) I call this Adam Smith’s second invisible hand, which concludes that profit maximizing capitalists can be relied on to choose new technologies in the social interest, i.e. laissez faire capitalism will be dynamically efficient; to distinguish it from Smith’s better known first invisible hand, which concludes that if left to themselves competitive markets will allocate scarce productive resources efficiently, i.e. laissez faire capitalism will achieve “static” efficiency. Whereas A.C. Pigou proved that Smith’s first invisible hand fails whenever there are externalities, Roemer’s theorem proves that Smith’s second invisible hand can fail whenever $r > 0$.

\(^3\) Proved as theorems 4.9 and 4.10 in Roemer (1981): 102-105.
is equal to \( V \), which means the private costs of inputs to capitalists are the same as the social costs of inputs to society. So when \( r = 0 \), and \( p = V \), when capitalists check if a new technology reduces their private costs of producing a good they are also checking if the new technology reduces the social costs of producing the good. But if the rate of profit is greater than zero, \( r > 0 \), then \( p \neq V \). Since capitalists use \( p \) to calculate the costs of inputs they may discover a new technology lowers their costs, and is therefore profitable, even when it raises the cost of production to society. And they may discover that a new technology raises their private costs, and is therefore not profitable, even though the new technology lowers the cost of production to society. This is why when \( r \) is greater than zero, and therefore \( p \neq V \), the kind of socially perverse phenomenon spelled out in Roemer’s theorem can occur.

Moreover, the higher the rate of profit, the more likely capitalist behavior is to be socially counterproductive: If the rate of profit is sufficiently high, and the wage rate therefore sufficiently low, capitalists will not implement a CU-LS change even if it is progressive. This is because when the wage rate is low enough the savings in labor costs will be less than the increase in capital costs -- making it non-viable -- no matter how large LS and how small CU may be. In the extreme, if \( r = r_{-\text{max}} \) and therefore \( w = 0 \), no CU-LS technical change will be viable and implemented no matter how much it reduces labor and how little it increases non-labor inputs, i.e. no matter how socially desirable it may be. Similarly, if the rate of profit is sufficiently high, and the wage rate therefore sufficiently low, capitalists will implement a CS-LU change even if it is retrogressive. This is because when the wage rate is low enough the additional labor costs will be less than the decrease in capital costs -- making it viable -- no matter how small CS and how large LU may be. Again, in the extreme if \( r = r_{-\text{max}} \) and therefore \( w = 0 \) every CS-LU change will be cost reducing no matter how much it increases labor inputs and how little it decreases non-labor inputs.

In sum, for any progressive CU-LS change, there is a rate of profit sufficiently high and corresponding wage rate sufficiently low to keep it from being

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4 The price equations for the economy are: \((1+r)pA + wL = p\). For \( r = 0 \): \( pA + wL = p \); \( p - pA = wL \); \( p[1-A] = wL \); \( p[1-A][1-A]^T = wL[1-A]^{-1} \); \( p = wL[1-A]^{-1} \); \( p = wV \), and \( p \) is proportional to \( V \).

5 For \( r > 0 \): \((1+r)pA + wL = p \); \( p - (1+r)pA = wL \); \( p[I - (1+r)A] = wL \); \( p[I - (1+r)A][I - (1+r)A]^T = wL[I - (1+r)A]^{-1} \); \( p = wL[I - (1+r)A]^{-1} \), and \( p \) is not proportional to \( V \).
viable. And for any retrogressive CS-LU change there is a rate of profit high enough and corresponding wage rate low enough to render it viable. Hence, the higher the rate of profit, and consequently the lower the wage rate, the more socially perverse the behavior of profit maximizing capitalists becomes as those we designate to make our technological choices for us. Notice there is never a problem with CS-LU progressive technologies which capitalists will always find viable no matter what r, w and p may be. Nor is there ever a problem with CU-LS retrogressive technologies which capitalists will never find viable no matter what r, w and p may be. Diagram 1 illustrates the problem.

Diagram 1

First we divide all technical changes into two disjoint sets – the set of all progressive technical changes (on the right), and the set of all retrogressive technical changes (on the left.) Next we eliminate all input saving technical changes from the progressive set since they are always viable and therefore never present a problem. Then we subdivide the set of progressive changes and the set of retrogressive changes into those that are CU-LS and those that are CS-LU as pictured. When r = 0 and therefore p = V the set of viable technical changes -- the hashed rectangle -- coincides exactly with the set of
progressive technical changes. However, as \( r \) rises and \( w \) falls this hashed set of viable technical changes moves leftward and starts to leave uncovered some of the CU-LS changes on the right of the progressive set -- leading to “sins of omission” -- and covers some of the CU-LS changes on the right of the regressive set -- leading to “sins of commission.” When \( r \) reaches its maximum and therefore \( w = 0 \) the set of viable technical changes -- the hashed rectangle -- includes only CS-LU changes in the middle portion of the diagram -- which means that in the extreme capitalists will fail to implement all progressive CU-LS changes and implement all retrogressive CU-LS changes.

3. The Okishio Theorem

*Okishio’s Theorem:* For \( A^* = [A+bL] \) non-negative, indecomposable, and productive, the introduction of any *viable* technological change must, necessarily *raise* the uniform rate of profit in the economy.\(^6\) The crucial step in the proof of the theorem is demonstrating that any viable technical change must, necessarily yield a value for the dominant eigenvalue for the new economy that is strictly lower than the dominant eigenvalue for the initial economy. In other words, \( \text{dom}(A^*) = \text{dom}(A' + bL') = \beta' < \beta = \text{dom}(A^*) = \text{dom}(A + bL) \), where \( \{A,L\} \) is the technology initially and \( \{A',L'\} \) is the technology after a viable change is adopted. Once this is established the result that the new uniform rate of profit in the economy, \( \rho' \), must be strictly greater than the old uniform rate of profit, \( \rho \), follows immediately. The genius of the theorem is that one cannot simply use the Frobenius-Perron theorem which establishes that if \( A^* \preceq A^* \) then \( \beta' < \beta \), because while some components of \( A^* \) must shrink for a change to be viable, others may well increase in size.

4. Resolving the Conundrum

If the change is retrogressive \( V' \succeq V \) and therefore \( V'b > Vb \). In other words, it now takes workers a larger fraction of every hour they work to produce the wage goods they consume per hour. This implies \( (1-V'b) < (1-Vb) \), which means that the fraction of every hour they work producing profits for their employer must fall. In which case, how is it possible for the rate of profit to rise, not fall, as the Okishio theorem proves it must?

\(^6\) The theorem was originally proved in Okishio (1961).
The answer to the conundrum lies in the fact that more hours will be worked to produce the same vector of gross outputs as before. Since the technical change is labor-using, \( L' \geq L \), it takes more hours of living labor to produce the same gross output vector as before, i.e. \( L'x > Lx \). If we think of employees (or hours worked) as lemons, in effect capitalists are not squeezing the same number of lemons as before for a given quantity of capital advanced. While they are not able to squeeze each lemon as strongly as they did before, \( (1-V'b) < (1-Vb) \), the number of lemons they are squeezing has increased, \( L'x > Lx \). This is how it is possible for the rate of profit to rise when the real wage and output stay constant even though the profit employers extract per hour of labor they hire has diminished.

5. The Dominant Eigenvalue and Productivity

However, resolving the conundrum suggests a further useful result: We can rigorously measure precisely how much any particular technical change, in any particular industry, increases the overall productivity of the economy – provided we are careful how we go about it.

Briefly returning to the conundrum: How can we reconcile the fact that the economy has become less efficient -- it takes us more hours to produce goods than before -- with the fact that the dominant eigenvalue of \( A^* \), \( \beta \), is necessarily made smaller by any viable technical change? A quick review of the proof of the Okishio theorem reveals that for \( A^* \) indecomposable, any viable technical change necessarily increases the rate of profit in the economy precisely because it diminishes \( \text{dom}(A^*) \), or \( \beta \). Isn’t the size of \( \beta \) a measure of the productivity of the economy, and a shrinking \( \beta \) a measure of increasing productivity? If a technical change is retrogressive it makes the economy less, not more productive. But if a retrogressive, CS-LU technical change is viable it is so precisely because it reduces, rather than increases \( \beta \).

The answer is that it depends on why \( \beta \) is shrinking. Recall that \( A^* = [A+bL] \). If \( \beta \) is shrinking because \( A \) is shrinking while \( L \) remains constant, this is a sign that the economy is becoming more productive. Or if \( \beta \) is shrinking because \( L \) is shrinking while \( A \) remains constant this is a sign that the economy is becoming more productive. But these are pure input saving technical changes, and clearly must increase the productivity of the economy and therefore raise the uniform rate of profit in the economy as long as the
real wage, $b$, remains constant. However, the difficult cases, which are also the more common cases, have always been those where a new technology reduces the amount of some input needed, but only by increasing use of some other input. Intuitively what we are searching for in such cases is a way to compare the size of the two effects. For a CS-LU change, is $A$ shrinking more than $L$ is increasing? In which case the CS-LU change would be progressive because it is increasing the productivity of the economy. For a CU-LS change, is $L$ shrinking more than $A$ is increasing? In which case the CU-LS change would be progressive because it is increasing the productivity of the economy. But notice that in the expression for $A^*$ the $L$ vector is weighted by the $b$ vector.

Consider what will happen to $\beta$ if some component of $L$ increases by a large amount while some component of $A$ decreases by only a little, but $b$ is very small, or in the extreme, equal to zero. With $b = 0$ any CS-LU technical change, no matter how much it expands $L$ and how little it diminishes $A$ will necessarily diminish $\beta$. But a smaller $\beta$ that results from this change is, by hypothesis, not due to an increase in the productivity of the economy. As a matter of fact, it quite obviously coincides with a decrease in economic productivity. So we must be careful when interpreting the meaning of changes in the dominant eigenvalue of $A^*$. Strictly speaking when $\text{dom}(A^*) = \beta$ shrinks it means the economy has become more profitable for employers – which is what the Okishio theorem proves -- but it does not necessarily mean the economy has become more productive -- as the above example demonstrates.

However, there are $b$’s that when used to “weigh” changes in $L$ to be compared with changes in $A$ will tell us whether or not the economy has become more productive. Recall that when $r = 0$ there is a one to one correspondence between viable and progressive technical changes. And for any $b$ sufficiently high so that the uniform rate of profit in the economy is reduced to zero we know: (1) $\text{dom}(A^*) = \text{dom}(A+b^L) = \beta = 1$. (2) Any technical change that is viable must reduce $\beta$ below 1 (Okishio). And (3) any technical change that is viable must also be progressive, or productivity enhancing (Roemer iii). Therefore, if $\text{dom}(A^{*-}) = \text{dom}(A'+b^L') = \beta' < 1$

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7 It is not necessary to resort to the Okishio theorem to prove that pure input saving technical changes must increase the rate of profit if the real wage is held constant. This can be proved using only the Frobenius-Perron theorem: If $A^* \leq A^*$ then $\beta' < \beta$ and $r' > r$. 

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the technical change must be progressive and increase productivity, while if \( \beta' > 1 \) it must be retrogressive and reduce productivity.

This provides a second method to determine if a technical change has increased or decreased the productivity of the economy. Instead of calculating and comparing the new and old \( V \)'s, we can simply see if \( \beta' \) is less than or greater than one -- where we have been careful to choose a \( \hat{b} \) so \( r = 0 \) initially. Moreover, this procedure has the advantage of telling us precisely how much economic productivity has increased, whereas comparing new and old \( V \)'s does not. Suppose, for example, we choose an appropriate \( \hat{b} \) so that initially \( \beta = 1 \). Then, using the same \( \hat{b} \) we calculate what the new dominant eigenvalue is after the technical change is introduced, and find that \( \beta' = 0.97 \). The increase in productivity is: \( \rho = (\beta - \beta')/\beta = (1.00 - .97)/1.00 = .03 \) or 3%.

To summarize: Changes in the size of \( \beta \), the dominant eigenvalue of any augmented input coefficient matrix \( A^* = A + bL \) produced by a technical change always tell us if the economy will become more profitable for employers provided the real wage, \( b \), remains constant. But changes in \( \beta \) only tell us if the economy will become more productive for \( b \)'s that correspond to an initial zero uniform rate of profit and \( \beta = 1 \). As demonstrated above, whether an economy that is more profitable for employers is also more productive from the perspective of its inhabitants may or may not prove to be the case for smaller \( b \)'s corresponding to initial rates of profit that are greater than zero, and therefore \( \beta \)'s that are less than one. We summarize these results as a theorem.

**Dominant eigenvalues, profitability, and productivity:** For any given \( A \) and \( L \) there are many \( A^* = [A+bL] \) corresponding to different real wage vectors, \( b \). For \( A^* \) non-negative, indecomposable, and productive:

(i) If a technical change reduces \( \text{dom}(A^*) = \beta \) for any \( A^* \), corresponding to any \( b \), then provided the real wage, \( b \), remains unchanged the uniform rate of profit in the economy, \( r \), must rise. (This part simply reiterates the Okishio theorem.)

(ii) For any \( \hat{b} \) sufficiently high to reduce the initial rate of profit in the economy to zero so that \( \text{dom}(A^*) = \beta = 1 \), if a technical change reduces the economy’s dominant eigenvalue, i.e. if \( \text{dom} (A^*) = \text{dom}(A'+b^\hat{L}') = \beta' < 1 \), it increases the productivity of the economy and is therefore “progressive”; if \( \beta' > 1 \) it decreases the productivity of the economy and is therefore
“retrogressive;” and if $\beta' = 1$ it does not affect the productivity of the economy and is therefore “neutral.”

(iii) The increase in the productivity of the economy is given by $\rho = (1 - \beta')$ where $\beta' = \text{dom}(A^*) = \text{dom}(A' + b^L)$.

References
