Magnetic field produced by a moving point charge

Exercise
Draw the vector MAGNETIC FIELD at the different points P, S, and L, respectively.
Magnetic field produced by a current

Strategy:
Divide the wire into small sections of length $\Delta l$

But notice,

$\Delta \vec{B} = \frac{\mu_0}{4\pi} \frac{\Delta q \vec{V} \times \vec{r}}{r^3}$

Contribution to the magnetic field due to the segment $\Delta \ell$ of the circuit is

$\Delta \vec{B} = \frac{\mu_0}{4\pi} \frac{I \Delta \ell \times \vec{r}}{r^3}$
Example: Magnetic field produced by a current that flows along a straight wire

\[ \Delta \mathbf{B} = \frac{\mu_0}{4\pi} \frac{I \Delta l \times \mathbf{r}^2}{r^3} \]

We want to evaluate the magnetic field at a point "P" located at a distance "R" from the wire.

For the case of an infinitely long wire:

\[ \Delta \mathbf{B} = \frac{\mu_0}{4\pi} \frac{I (\Delta l) R}{r^3} \]

or \[ r \sin \theta = R \]

\[ r^3 = (R^2 + R^2)^{3/2} \]

Contribution to the MAGNETIC FIELD at P from just one current segment \( \Delta Z \)

\[ \mathbf{B} = \frac{\mu_0 I R}{4\pi} \int_{-\infty}^{\infty} \frac{\Delta Z}{(z^2 + R^2)^{3/2}} \]
\[ B = \frac{\mu_0 I R}{4\pi R^2} \left( \frac{e^{-\frac{z}{R}}}{\sqrt{z^2 + R^2}} \right) \]

Notice:
\[ \lim_{z \to \infty} \frac{z}{\sqrt{z^2 + R^2}} = +1 \]
\[ \lim_{z \to -\infty} \frac{z}{\sqrt{z^2 + R^2}} = -1 \]

\[ B = \frac{\mu_0 I}{2\pi R} \]

Lateral View:

Infinitely long wire:

\[ B_P = B_Q \]
Question:

\[ B < B' \quad B = B' \quad B > B' \]
Question: What is the contribution to the magnetic field at P from the finite wire segment AB of length L?

\[ B = \frac{\mu_0 I R}{4\pi} \int_{-L/2}^{L/2} \frac{dz}{(z^2 + R^2)^{3/2}} \]

Question: What is the magnetic field at the point P caused by the segment AB that carries a current I?

\[ B = \frac{\mu_0 I R}{4\pi} \int_{a}^{b} \frac{dz}{(z^2 + R^2)^{3/2}} \]
\[
\int_{-a}^{b} \frac{dz}{(z^2 + r^2)^{3/2}}
\]

Change of variables

\[Z = R \tan \alpha \]

\[Z^2 = R^2 \tan^2 \alpha \]

\[Z^2 + r^2 = R^2 (\tan^2 \alpha + 1) = R^2 \sec^2 \alpha \]

\[(Z^2 + r^2)^{3/2} = R^3 \sec^3 \alpha \]

\[dz = R \, d(\tan \alpha) = R(\sec^2 \alpha) \, d\alpha \]

\[
\int_{-a}^{b} \frac{dz}{(z^2 + r^2)^{3/2}} = \int_{\theta_a}^{\theta_b} \frac{\sec^2 \alpha \, d\alpha}{R^3 \sec^3 \alpha} = \frac{d\alpha}{R^2 \sec \alpha}
\]

\[
= \left[ \frac{R}{R^2} \frac{\cos \alpha}{\sqrt{b^2 + r^2}} \right]_{\theta_a}^{\theta_b}
\]

\[
= \left( \frac{1}{R^2} \frac{b}{\sqrt{b^2 + r^2}} \right) - \left( \frac{1}{R^2} \frac{-a}{\sqrt{a^2 + r^2}} \right) = \frac{1}{R^2} \left( \frac{b}{\sqrt{b^2 + r^2}} + \frac{a}{\sqrt{a^2 + r^2}} \right)
\]
Exercise: Calculate the magnetic field at the points "G", "H", and "K" produced by the 2-meter wire that carries a current $I = 0.5$ Amps.
Force between two parallel currents

We are assuming the wires are infinitely long.

The magnetic field $B_1$ produced by the current $I_1$ at the site 2 is,

$$B_1 = \frac{\mu_0 I_1}{2\pi R}$$

- A segment $L$ of wire-2 is immersed in a magnetic field $B_1$. So the wire will experience a force,

$$\vec{F}_{21} = I \cdot \vec{I}_2 \times \vec{B}_1$$

$$B_1 = \frac{\mu_0 I_1}{2\pi R}$$
\[ \vec{F}_{21} = -F_{21} \hat{z} \quad \text{(attractive force)} \]

where \[ F_{21} = I_2 l_0 \frac{\mu_0 I_1}{2\pi R} \]

The longer the segment \( l_0 \), the stronger the force.

The magnetic field produced by an infinitely long wire carrying a current \( I_1 \)

current flowing through wire \( 2 \)

force per unit length \[ \frac{F_{21}}{l_0} = \frac{\mu_0 I_1 I_2}{2\pi R} \]

Notice the symmetry of the result with respect to the currents \( I_1 \) and \( I_2 \).
The symmetry of the previous result indicates that
\[ \frac{F_{12}}{L} = \frac{\mu_0}{2\pi R} I_2 I_1 \]
and \[ \vec{F}_{12} = F_{12} \hat{z} \]

Parallel currents attract each other
Antiparallel currents repel each other
The magnetic dipole moment

\[ \vec{\tau} = \vec{\mu} \times \vec{B} \]

\[ \vec{\mu} = \vec{i} \times \text{area} \]

The torque \( \tau \) makes the dipole \( \mu \) to have a tendency to be aligned along the external magnetic field.

**Magnetic potential energy of a dipole immersed in an external magnetic field**

As usual, the potential energy of the dipole will be given with respect to a configuration reference.

\[ \theta_{net} = \frac{\pi}{2} \]

How much external work is needed to take the loop from the configuration-1 to the configuration-2?
\[ \mathbf{F}_{\text{ext}} = i l B \sin \theta \]

\[ W_{\text{ext}} = - \left\{ \int_{\Psi / 2}^{\Psi / 2} |\mathbf{F}_{\text{ext}}| \, ds \right\} = - \int_{\Psi / 2}^{\Psi / 2} |i l B \sin \theta| \, ds = - \int_{0}^{\Psi / 2} i l B \sin \theta \frac{d\theta}{2} \]

\[ = - i l B \frac{\Psi}{2} \int_{\Psi / 2}^{\Psi / 2} \sin \theta \, d\theta = - i l B \frac{\Psi}{2} \cos \theta \]

\[ - \cos \theta \bigg|_{\Psi / 2} = (0) - (-1) = \cos \theta \]

The total external work will be twice this value.

\[ W_{\text{ext}} = -i l B \mu \cos \sigma \stackrel{\text{total}}{=} - \mathbf{B} \cdot \mathbf{A} \cos \sigma = - B \mu \cos \sigma \]

\[ W_{\text{ext}} = - \mu \cdot \mathbf{B} \]
\[ \mu = N i A \\
= 20 \times (0.1 \text{ Amp}) \times (10^{-1}\text{m}) \times (5 \times 10^{-2}\text{m}) \\
= -10^{-2}\text{ A-m}^2 \hat{k} \]

\[ B = 0.5\text{ T} \]

\[ \vec{\tau} = \vec{\mu} \times \vec{B} \]

\[ \vec{\tau} = (-j) 10^{-2} \times 0.5 \times \frac{\sqrt{3}}{2} \]

\[ = -0.42 \times 10^{-2} j \]

\[ = -4.2 \times 10^{-3} j \]
Definition of the unit current: The Ampere

\[ F = \frac{M_0}{4\pi} \frac{2}{R} \]

The ampere is that constant current which, if maintained in two straight parallel conductors, placed one meter apart, the conductor will exert a force on each other of \(2 \times 10^{-7}\) newtons per meter of length.

If \( R = 1 \text{ m} \) and \( \frac{E}{L} = 2 \times 10^{-7} \text{ newtons} \)

then \( I = 1 \text{ ampere} \).
Having defined the **ampere** (unit of current) and knowing that \( i = \frac{q}{t} \), the unit of change is defined as follows:

The **coulomb** (C) is the amount of change flowing through a wire in \( t = 1 \text{ sec} \) when the current in wire is 1 ampere.

\[ \text{coulomb} = 1 \text{ ampere \cdot second} \]
Along the segment BC the magnetic field produced by the 30 Amp wire is constant.

Also, along the segment DA the magnetic field produced by the 30 Amp wire is constant.

For these two cases, it is convenient to use the expression,

$$\vec{F} = I \vec{l} \times \vec{B}$$

Along the segment AB the magnetic field produced by the 30 Amp wire varies with position. Hence, in order to calculate the force on that segment of the loop, it is better to use,

$$d\vec{F} = I (d\vec{l}) \times \vec{B}$$
Calculation of the force on the segment AB

The magnitude of the magnetic field at the point of coordinate \( z \) is given by

\[
B(z) = \frac{\mu_0 I_1}{2\pi} \frac{1}{(9 \times 10^{-2} m - z)}
\]  

(1)

The vector magnetic field is given by

\[
\vec{B}(z) = B(z) \hat{j}
\]

(2)

The force acting on the segment \( \vec{dl} \) = \((dz) \hat{k} \) is,

\[
\Delta F = I_2 \vec{dl} \times \vec{B}(z)
\]

\[
= I_2 [(dz) \hat{k}] \times [B(z) \hat{j}]
\]

\[
= I_2 (dz) B(z) \hat{k} \times \hat{j}
\]

\[
= I_2 (dz) B(z) (-\hat{i})
\]

(3)
\[ \Delta F = - \left[ I_2 B(z) \, dz \right] \hat{z} \] 

Working out the magnitude of the force.

\[ I_n (4) \]

\[ \Delta F = I_2 B(z) \, dz \]

\[ F = \int_{AB} I_2 B(z) \, dz \]

\[ = I_2 \int_{A}^{B} B(z) \, dz = I_2 \int_{z=0}^{z=8 \times 10^{-2}} \frac{\mu_0 I_1}{2\pi} \frac{dz}{(9 \times 10^{-2} \, m - z)} \]

\[ = \frac{\mu_0 I_2 I_1}{2\pi} \left[ \ln \left( \frac{9 \times 10^{-2}}{1 \times 10^{-2}} \right) \right] \]

\[ = \frac{\mu_0 I_2 I_1}{2\pi} \ln \left( \frac{9 \times 10^{-2}}{1 \times 10^{-2}} \right) = \frac{\mu_0 I_2 I_1}{2\pi} \ln g = F \]

\[ F_{AB} = - \frac{F}{\lambda} \]

\[ \vec{F}_{AB} = - \frac{F}{\lambda} \hat{\lambda} \]
Calculation

\[ F_{AB} = \frac{4I_1 I_2 m m}{2\pi} \ln \left( \frac{R}{r} \right) \]

\[ = 2 \cdot 4 \times 10^{-20} (20) (30) \ln 9 = 1.2 m \times 10^{-7} \text{ Newtons} \]

\[ = 2.6 \times 10^{-9} \text{ Newtons} \]
Example

KL is a semi-circle of radius "R"

What is the magnetic field at point "P"?

Solution

Contribution to the magnetic field at P from the section (-∞ to K)?

ΔB = 0

Contribution to the magnetic field at P from the section (L to ∞)?

ΔB = 0

Contribution to the magnetic field at P from the semicircle KL?

Along the circular path, what is the angle between Δl and r? 90 degrees.

What will be the direction of the total magnetic field? Inside the plane of the figure.

\[ \Delta B = \frac{\mu_0 I}{4\pi} \left| \Delta l \times r \right| = \frac{\mu_0 I}{4\pi} \frac{(\Delta l) R}{r^3} = \frac{\mu_0 I}{4\pi} \frac{\Delta l}{R^2} \]
\[ B = \frac{\mu_0 I}{4\pi} \int \frac{\Delta l}{R^2} = \frac{\mu_0 I}{4\pi} \frac{1}{R^2} \int \Delta l = \]

\[ = \frac{\mu_0 I}{4\pi} \frac{1}{R^2} \pi R = \frac{\mu_0 I}{4R} \]

Magnitude of the magnetic field at point P

Example: Circular arc

Following a similar procedure from the previous example we obtain.

\[ B = \frac{\mu_0 I}{4\pi} \frac{\phi}{R^2} \]

magnetic field at the centre of the arc

or, equivalent

\[ B = \frac{\mu_0 I}{4\pi} \frac{\Theta}{R} \quad \Theta \text{ in radians} \]

Exercise: Find the magnetic field at point "P" (the centre of the arc of radius R)
Exercise: Sketch the magnetic field of a circular coil of radius $R$ carrying a current $I$. 
Magnetic field lines established by two loops.
Region of weaker magnetic field

Region of stronger magnetic field

Region of weaker magnetic field

Region of stronger magnetic field

South magnetic pole

North magnetic pole
Question: In the figure above, where is the magnitude of the magnetic field higher, at point P or at point Q?

Notation: 
\[ n = \frac{N}{L} \]

= # of turns per unit length
The Earth as a big magnetic dipole

What are the orientation of the magnetic field lines?
The Earth as a Big Magnetic Dipole

Earth's Magnetic Field:
- **North Magnetic Pole**
- **South Magnetic Pole**
- **EQUATOR**
- **Magnetic Axis**
- **Axis of Rotation**
- **12°**

Earth's surface field:
- $B \approx 10^{-4} T$
- $1$ Gauss

What are the orientation of the magnetic field lines?
The magnetic field of the Earth has undergone reversals of magnetic polarity

Every few million years

Age of Universe $= 1.6 \times 10^9$ million years
$= (5 \times 10^3$ seconds)