CAPACITANCE

Parallel-plates capacitor

\[ V_2 - V_1 = - \int E \, dx \]

\( E \) is assumed to be uniform between the plates

\[ = -E \int_1^2 dx = -E d \]

\[ V_2 - V_1 = -E d \]

This expression indicates that \( V_1 > V_2 \)

On the other hand, we have an alternative expression for \( E \) applicable to this particular charge arrangement:

\[ E = \frac{F}{\varepsilon_0} = \frac{Q/A}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A} \]

\[ V_1 - V_2 = E d \]

\( V \) (Voltage across the battery)

\[ V = E d = \frac{Q}{\varepsilon_0 A} d \]

\[ \frac{\varepsilon_0 A}{d} V = Q \]

Capacitance

\[ Q = \varepsilon_0 V \]

Capacitance
Question

\[ E_1 < E_3 \]
\[ E_1 = E_3 \]
\[ E_1 > E_3 \]

\[ d_3 > d_1 \]
Question

\[ A_2 > A_1 \]

\begin{align*}
E_1 &< E_3 \\
E_1 &= E_3 \\
E_1 &> E_3
\end{align*}
We use capacitors to store charge:

**Step-1**
Frontal view of two parallel plates
Each plate is called an "electrode"

**Step-2**
Connection to a battery

**Step-3**
After removing the batteries the plates remain charged

Notice: The higher the voltage, the more charge is deposited in the electrodes. $Q = CV$
Capacitors also store energy

The question we are going to answer is "How much energy is spent in charging a capacitor?"

Step-1
Uncharged plates
\[ C = \varepsilon_0 \frac{A}{d} \]

Step-2
Imagine, we take an amount of charge \( \Delta q \) from the right plate and deposit it into the left plate. We repeat this process again and again.

Step-3
Assume, at a given time, an amount of charge \( q \) has been transferred from the right plate to the left plate. The potential difference between the plates is \( V = \frac{q}{C} \)

Step-4
External work required to transport a small charge \( \Delta q \) from the right plate to the left plate

\[ \Delta W_{\text{ext}} = \int_{-\Delta q}^{\Delta q} E \cdot d = \left( \frac{\Delta q}{2} \right) E \cdot d = \left( \frac{\Delta q}{2} \right) V \]

External work required to transport a small charge \( \Delta q \) from the right plate to the left plate

\[ \Delta W_{\text{ext}} = V \Delta q = \frac{q}{C} \Delta q \]
\[ W_{\text{ext}} = \int_{0}^{a} \frac{9}{2} \, dq \mid_{0}^{a} = \frac{1}{2} \frac{9}{2} a^2 \]

\[ W_{\text{ext}} = \frac{1}{2} \frac{Q^2}{\varepsilon} \]

External energy required to change a capacitor.

And, since \( Q = C V \)

\[ W_{\text{ext}} = \frac{1}{2} CV^2 \]

More common notation

\[ U = \frac{1}{2} \frac{q^2}{\varepsilon} = \frac{1}{2} CV^2 \]

\[ U \text{ : Energy stored in the capacitor.} \]

Question: Where is this energy \( U \) located?

Notice

before \[ \begin{array}{c}
\end{array} \]

after \[ \begin{array}{c}
\end{array} \]
We may say that the energy $U$ is distributed inside the volume of the plate capacitor.

We can further say that the energy $U$ is stored in the electric field $E$.

Some hints:

$$U = \frac{1}{2} CV^2 = \frac{1}{2} C (Ed)^2 = \frac{1}{2} \varepsilon_0 \frac{A}{d} \cdot (Ed)^2$$

$$= \frac{1}{2} \varepsilon_0 \frac{A}{d} E^2$$

$$\frac{U}{\text{volume}} = \frac{U}{Ad} = \frac{1}{2} \varepsilon_0 E^2$$

wherever there exists an electric field $E$, there exist a potential energy density $\mu$

$$\mu = \frac{1}{2} \varepsilon_0 E^2$$

potential energy per unit volume

"The electric potential energy stored in the capacitor is considered to be stored in the electric field"
Several capacitors connected in SERIES

\[ V = V_1 + V_2 + V_3 \]

\[ V = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} \]

\[ V = Q \left( \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \right) \]

\[ \Rightarrow \frac{1}{C_{\text{equiv}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \]

\[ Q = C_{\text{equiv}} \cdot V \]

\[ C_{\text{equiv}} = \left( \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \right)^{-1} \]
where $C_1 \neq C_2 \neq C_3$

Zero net charge in this section

Question:

Is $E = E'$?

Is $E' = E''$?
Several capacitors connected in PARALLEL

\[ q_1 = C_3 V \quad q_2 = C_2 V \quad q_3 = C_1 V \]

Therefore:

\[ Q = q_1 + q_2 + q_3 = C_1 V + C_2 V + C_3 V = \frac{V}{C_{\text{equiv}}} \]

On the other hand, \( Q = V C_{\text{equiv}} \) which implies,

\[ C_{\text{equiv}} = C_1 + C_2 + C_3 \]
QUESTION: In the particular case that $d_1 = d_2 = d_3$, what can we say about $E_1$, $E_2$, and $E_3$?

$E_3 = E_2$ ?

$E_2 = E_1$ ?

(assume $C_1 \neq C_2 \neq C_3$)
Example

**Step 1**

Find how much charge is stored in the capacitor.

$q_0 = ? = C_1 V_0$

**Step 2** Battery Removed

What is the potential difference across the capacitor?

$V_0 = ? \quad 6.3 \text{ volts}$

**Step 3**

What is the potential difference across the capacitors?

$C_1 = 3.55 \mu F$

$C_2 = 8.95 \mu F$
Charge conservation indicates

\[ q_0 = q_1 + q_2 \]

before \hspace{2cm} after

\[ C_1 V_0 = C_1 V_{\text{new}} + C_2 V_{\text{new}} \]

\[ \frac{C_1 V_0}{C_1 + C_2} = V_{\text{new}} \]

\[ V_{\text{new}} = 1.79 \text{ V} \]

From step 2

\[ \begin{align*}
q_0 &= C_1 \quad V_0 \quad \text{ Potential Energy Stored} \\
-q_0 &= C_2
\end{align*} \]

From step 3

\[ \begin{align*}
U_C &= C_1 \quad V_{\text{new}} \quad \text{ Potential Energy Stored} \\
U_C &= C_2
\end{align*} \]

Is there a violation of the conservation of energy in this example?
\[ U_i = \frac{1}{2} \frac{q_0^2}{C_1} = \frac{1}{2} \frac{(22.4 \text{ \mu C})^2}{3.55 \text{ \mu F}} = 70.45 \text{ \mu J} \]

\[ U_f = \frac{1}{2} \frac{q_1^2}{C_1} + \frac{1}{2} \frac{q_2^2}{C_2} \]

\[ = \frac{1}{2} \frac{(C_1 V_{\text{new}})^2}{C_1} + \frac{1}{2} \frac{(C_2 V_{\text{new}})^2}{C_2} \]

\[ = \frac{1}{2} \frac{(6.35 \text{ \mu C})^2}{C_1} + \frac{1}{2} \frac{(16 \text{ \mu C})^2}{C_2} \]

\[ = 5.69 \text{ \mu J} + 14.34 \text{ \mu J} \]

Notice

\[ U_f < U_i \]

final initial
Extracurricular reading:
Feynman Lectures on Physics, Vol. II
"The ambiguity of the field energy" , page 27-6

- No unique way to define \( U \) neither whose "exactly" is located.

- Possible solution: theory of gravitation
  Theory of gravity says:
  "all energy is source of gravitational attraction"

Therefore, this theory will require that,

The energy density of electricity must be located properly is we are to know in which direction the gravity force acts.

Position of the star

Light bends as it passes near the Sun

Observed position of the star
Extra curricular reading:

- *Feynman Lectures, Vol-II*, chapter 28
  "ELECTROMAGNETIC MASS"
Capacitors with a dielectric

The capacitor was initially charged using a battery. Then the battery was removed.

Inserting a dielectric:

(no batteries connected to the plates. Hence, the charge on the plates remains constant).

\[ C_0 = \frac{\varepsilon_0 A}{d} \]

\[ V_0 = E_0 d \]

Experimentally it is found that the potential difference between the plates decreases.

\[ V = \frac{V_0}{K} \]

K is called the dielectric constant (It is a material property)
what is the new electric field \( E \)?

Since \( V = Ed \) we obtain

\[
\frac{V_0}{k} = Ed
\]

\[
E_0 \cdot \frac{V_0}{k'd'} = E \quad \Rightarrow \quad E = \frac{E_0}{k}
\]

what is the new capacitance

\[
C = \frac{\text{charge}}{\text{voltage}}
\]

\[
= \frac{Q_0}{V_0 / k} = k \frac{Q_0}{V_0}
\]

\[
C = k \, C_0
\]
Example

A parallel-plate capacitor is initially connected to a battery.

\[ q = CV = 13.5 \times 10^{-12} \text{ F} \times 12.5 \text{ volts} = 1.68 \times 10^{-10} \text{ C} \]

b) The battery is disconnected and a dielectric \((k = 6.5)\) is slipped between the plates.

\[ q = 1.68 \times 10^{-10} \text{ C} \]

What are the energies stored in the capacitor before and after introducing the dielectric?

<table>
<thead>
<tr>
<th></th>
<th>Before</th>
<th>After</th>
</tr>
</thead>
<tbody>
<tr>
<td>(U_i)</td>
<td>[ U_i = \frac{1}{2} \frac{q^2}{C} = \frac{1}{2} \frac{(1.68 \times 10^{-10} \text{ C})^2}{13.5 \times 10^{-12} \text{ F}} ]</td>
<td>[ U_f = \frac{1}{2} \frac{q^2}{k \varepsilon_0} ]</td>
</tr>
<tr>
<td></td>
<td>[ = 1055 \times 10^{-12} \text{ J} ]</td>
<td>[ = 162 \times 10^{-12} \text{ J} ]</td>
</tr>
</tbody>
</table>
Notice, the electrical energy decreases by a factor $K = 6.5$ after the dielectric is introduced.

"the capacitor exerts a tiny tug on the slab"

dielectric slab

Example: consider the previous problem but keeping the battery connected

$Q = 1.6 \times 10^{-9}$
$c = 13.5 \mu F$

$\text{Is } Q = q$?
$q < q$?
$q > q$?
12.5 volts $= E \cdot d$  

So, the electric field has to be the same in both cases.

We also know a relationship between the electric field and the amount of charge on the plates

$$E = \frac{q/A}{\varepsilon_0}$$  

Therefore:

$$q = \frac{Q}{K}$$

or $$Q = K \cdot q$$
DIELECTRICS: A microscopic View

Metallic plate

Free charge

Neutral region

Material of dielectric constant K

Metallic plate

Free charge

Electric field due to q and -q

Electric field due to q' and -q'

Equivalent
Let's use Gauss' Law to find out the induced charge $q'$.

Metallic plate

Gaussian surface

Material of dielectric constant $K$

The charge inside the Gaussian surface is $q - q'$. 

$$\varepsilon_0 \int \vec{E} \cdot d\vec{s} = \text{Charge inside the Gaussian surface} = q - q'$$

$$\varepsilon_0 EA = q - q'$$

$$\varepsilon_0 \frac{E_0}{K} A = q - q'$$
But, let’s remember that

\[ E_0 = \frac{\mathcal{V}}{E_0} = \frac{q}{A\varepsilon_0} \]

value of the electric field without dielectric inside the plates.

So, replacing (3) in (2) we obtain

\[ \frac{q}{k} = q - q' \]

Or

\[ q' = \frac{k-1}{k} q \]

Notice also that, replacing (4) in (1) we obtain

\[ \oint \varepsilon_0 k E \cdot d\mathbf{s} = q \]

Gauss’ Law applied to dielectric materials
Example

Parallel-plate capacitor filled with 2 dielectrics
Find the capacitance of the device.

Solution: The device looks like two capacitors connected in parallel.

\[ C_1 = K_1 \frac{(A/2) E_0}{d} \]
\[ C_2 = K_2 \frac{(A/2) E_0}{d} \]

\[ C = C_1 + C_2 \quad \text{(for parallel connection)} \]
\[ = \frac{(A/2) E_0}{d} \left( K_1 + K_2 \right) \]

\[ C = \frac{AE_0}{d} = \frac{K_1 + K_2}{2} \]
Example

Parallel-plate capacitors filled with 2 dielectrics. Find the capacitance of the device.

Solution: It is a little less obvious in this case that we have the equivalent of 2 capacitors connected in series.

So, let's follow a more secure method to solve this problem.

\[
V = E_1 \cdot \frac{d}{2} + E_2 \cdot \frac{d}{2}
\]

\[
= \frac{E_0}{K_1} \cdot \frac{d}{2} + \frac{E_0}{K_2} \cdot \frac{d}{2}
\]

\[
= \frac{E_0}{2} \left( \frac{1}{K_1} + \frac{1}{K_2} \right)
\]

but \( E_0 = \frac{Q}{\epsilon_0} = \frac{Q}{A \epsilon_0} \)

\[
= \frac{Q d}{A \epsilon_0} \left( \frac{1}{K_1} + \frac{1}{K_2} \right)
\]
\[ Q = \frac{A \varepsilon_0}{d} \left( \frac{2 k_1 k_2}{k_1 + k_2} \right) V \]
Example

Find the charge distribution in the plates.

Plate of area $A$

Find the capacitance.

\[
\begin{align*}
V &= E_1 \cdot d \\
V &= E_2 \cdot d \\
\Rightarrow & \quad E_1 = E_2
\end{align*}
\]

Since $K_1 \neq K_2$ and $E_1 = E_2$ then we suspect that the charge distribution is not uniform over the plates.

What is the relationship between the charges?
\[ E_1 = \frac{\text{Field without dielectric}}{k_1} = \frac{q_1}{(A/2)\varepsilon_0} = \frac{2q_1}{AE_0k_1} \]

\[ E_2 = \frac{\text{field without dielectric}}{k_2} = \frac{2q_2}{AE_0k_2} \]

\[ E_1 = E_2 \Rightarrow \frac{q_1}{k_1} = \frac{q_2}{k_2} \quad (1) \]

\[ Q = (q_1 + q_2) = q_1 + \frac{k_2}{k_1} q_1 \]

\[ = q_1 \left(1 + \frac{k_2}{k_1}\right) \quad (2) \]
But \( Q_1 = C_1 V \) where \( C_1 = \frac{(A/2) \epsilon_0}{d} k_1 \)

\[
Q_1 = \frac{A \epsilon_0 k_1 V}{2d} \tag{3}
\]

So, replacing (3) in (2) we obtain

\[
Q = \frac{A \epsilon_0 k_1 (1 + \frac{k_2}{k_1}) V}{2d}
\]

\[
C = \frac{A \epsilon_0}{2d} (k_1 + k_2)
\]
What is ground?

It is a very large conductor that can supply unlimited amount of charge.

**Review on Dielectric Breakdown:**

Many nonconducting materials become ionized in very high electric fields $E$. This phenomenon is called *dielectric breakdown*. A diagram illustrating the breakdown can be seen here, with a focus on the interaction between the electric field $E$ and the dielectric material, which is depicted as containing dipoles and atoms under intense electric fields.