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Rays and Optical beams

In geometrical optics, the propagation of optical waves can be described approximately by using the concept of rays. This is valid provided the beam diameter is much larger than the wavelength and the diffraction can be neglected. The rays travel in straight lines in homogeneous media and obey Fermat’s principle of least time in inhomogeneous media.

A good understanding of the propagation of rays makes possible to trace their trajectory when they are passing through various optical media. One finds that the passage through these optical elements can be described by simply 2x2 matrices.

This section also describes the propagation of Gaussian beams in various optical media. It will be shown that simple ray matrices can be employed to describe the propagation of spherical waves and Gaussian beams, such as those characteristic of the output of lasers.

I. Ray Matrices

We will consider the case of:

**Paraxial rays** through optical systems.

Optical systems with an axis of symmetry (z-axis) also called the optical axis

Paraxial rays are those whose inclination (θ) relative to the optical axis is small such that

\[ \sin(\theta) \sim \tan(\theta) \sim \theta \]

We will restrict ourselves also to the case of **meridional rays**, i.e. each ray trajectory is confined in a plane that passes through the optical axis.

Given the cylindrical symmetry that we are considering, the trajectories can be conveniently described by \( r(z) \), where \( r \) is the distance measured from the optical axis.

I.A Special cases of ray matrices

Case: Propagation through a thin lens

From Fig. 1, we obtain,

\[ r_{\text{out}} = r_{\text{in}} \]

ray position does not changed; this follows from the assumption of the thin lens.

From Fig. 1, \( r_{\text{out}}' = \frac{f r_{\text{in}}' - r_{\text{in}}}{f} \), which gives,

\[ r_{\text{out}}' = r_{\text{in}}' - \left( \frac{r_{\text{in}}}{f} \right) \]

the lens changes the slope of the incident ray.

These relationships can be conveniently expressed using a 2x2 matrix

\[
\begin{bmatrix}
0 & 1 \\
-1 & f \\
\end{bmatrix}
\begin{bmatrix}
r_{\text{in}} \\
r_{\text{in}}' \\
\end{bmatrix}
= 
\begin{bmatrix}
r_{\text{out}} \\
r_{\text{out}}' \\
\end{bmatrix}
\]

**deflection by a thin lens**

(1)
Case: Propagation through an homogeneous medium
Similarly, the propagation though a homogeneous medium with length \( d \) will be given by,

\[
\begin{align*}
    r_{\text{out}} &= r_{\text{in}} + d \, r'_{\text{in}} \\
    r'_{\text{out}} &= r'_{\text{in}}
\end{align*}
\]

Fig. 2 Ray propagation through an homogeneous medium.

\[
\begin{bmatrix}
    r_{\text{out}} \\
    r'_{\text{out}}
\end{bmatrix} =
\begin{bmatrix}
    1 & d \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    r_{\text{in}} \\
    r'_{\text{in}}
\end{bmatrix}
\]

\text{ propagation through homogeneous medium of length } d

Case: Reflection from a spherical mirror of radius R_2. Incident ray travelling from left to right.

Fig. 3 Ray travelling from left to right, reflecting from a spherical mirror of radius R_2. The right side shows the interpretation of \( r' \) (i.e. the slope of the corresponding rays) as indicated in expression (3). For convenience we will modify this expression, as explained in the note below expression (3) and in Fig. 4
It is left as an exercise to show that for the reflection shown in Fig. 3 one obtains

\[
\begin{bmatrix}
  r_{\text{out}} \\
  r'_{\text{out}}
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 \\
  2/ R_2 & -1
\end{bmatrix}
\begin{bmatrix}
  r_{\text{in}} \\
  r'_{\text{in}}
\end{bmatrix}
\]

(3)

However, since the travelling direction of the ray is reversed after the reflection, a more convenient expression for the 2x2 matrix is considered below.

- In the expression above, \(r'_{\text{out}}\) stands for the slope of the output ray (see Fig. 3). Notice, however, that if we considered a subsequent ray propagation through an homogeneous medium then we would have to use the matrix (2) but with a negative value for \(d\).

- To avoid such an inconvenience, one can replace the value of \(r'_{\text{out}} = 2/R_2 \ r_{\text{in}} - r'_{\text{in}}\) in expression (3) by a similar expression but with a negative sign: \(- (2/R_2 \ r_{\text{in}} - r'_{\text{in}})\).
  That way, \(- (2/R_2 \ r_{\text{in}} - r'_{\text{in}})\) d (with positive d) would give the change in the ray position upon reflection. For this reason, instead of (3), the reflection by a mirror is preferable expressed as,

\[
\begin{bmatrix}
  r_{\text{out}} \\
  r'_{\text{out}}
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 \\
  -2/ R_2 & 1
\end{bmatrix}
\begin{bmatrix}
  r_{\text{in}} \\
  r'_{\text{in}}
\end{bmatrix}
\]

(4)

Conveniently, in this expression we are using \(r'_{\text{out}}\) as the negative value of the slope of the reflected output ray.
That way in a subsequent application of a free propagation, the corresponding 2x2 matrix (2) can be used with a positive value for \(d\).

**Case: Reflection from a spherical mirror of radius \(R_2\). Incident ray travelling from right to left.**

For the case of an incident ray travelling from left to right, we made an additional convenient modification to the 2x2 matrix (3) to take into that a subsequent propagation of the ray is in the negative direction; this resulted in the 2x2 matrix (4). One may wonder what would be the 2x2 matrix corresponding for the case in which the incident ray travels from right to left. How different would it be this case compared to expression (4)?
Fig. 4 Reflection from a spherical mirror (incident ray travelling from right to left).

Solution

Following the convention used for rays travelling from left to right, we expect that the value of $r'_{in}$ shown in the figure above is equal to the negative of the slope of the incident ray; i.e.

$$r'_{in} = -\tan \theta$$  \hspace{1cm} (5)

in the paraxial approximation,

$$r'_{in} = -\theta$$  \hspace{1cm} (5')

From Fig. 5  \hspace{0.5cm} \alpha = \beta + \theta

$$\phi = \alpha + \beta$$

This gives, $\phi = \alpha + \beta = \alpha + (\alpha - \theta)$

$$\phi = 2\alpha - \theta$$  \hspace{1cm} (6)

Notice also in Fig. 5 that (within the paraxial approximation),

$$\alpha = \sin \alpha = -\frac{r_{in}}{R_1}$$  \hspace{1cm} (7)

(the negative sign is because, in the figure, the reflection occurs below the optical axis and the angle $\alpha$ in the graph is positive).

Replacing (5)' and (7) in (6),

$$\phi = 2\alpha - \theta$$

$$= 2 \left( -\frac{r_{in}}{R_1} \right) - ( -r'_{in}) = -2 \left( \frac{r_{in}}{R_1} \right) + r'_{in}$$

Since $\phi = \tan \phi = r'_{out}$,
\[
r'_{\text{out}} = -2 \left( \frac{r_{\text{in}}}{R_1} \right) + r'_{\text{in}} \tag{8}
\]

Also, a reflection in the mirror gives,

\[
r_{\text{out}} = r_{\text{in}} \tag{9}
\]

The results (8) and (9) are written in a more compact form,

\[
\begin{bmatrix}
    r'_{\text{out}} \\
    r''_{\text{out}}
\end{bmatrix} = \begin{bmatrix}
    1 & 0 \\
    -\frac{2}{R_1} & 1
\end{bmatrix}
\begin{bmatrix}
    r_{\text{in}} \\
    r'_{\text{in}}
\end{bmatrix} \tag{10}
\]

Thus, for the case of a deflection of a ray arriving from the right side and incident on a spherical mirror of radius of curvature \( R_1 \), one obtains,

\[
\begin{bmatrix}
    r'_{\text{out}} \\
    r''_{\text{out}}
\end{bmatrix} = \begin{bmatrix}
    1 & 0 \\
    -\frac{2}{R_1} & 1
\end{bmatrix}
\begin{bmatrix}
    r_{\text{in}} \\
    r'_{\text{in}}
\end{bmatrix}
\]

Fig. 6 Reflection from a spherical mirror

**Equivalence between the reflection from a spherical mirror and the refraction through a thin lens.**

A comparison between expression (1) \( \begin{bmatrix}
    r'_{\text{out}} \\
    r''_{\text{out}}
\end{bmatrix} = \begin{bmatrix}
    1 & 0 \\
    \frac{1}{f} & 1
\end{bmatrix}
\begin{bmatrix}
    r_{\text{in}} \\
    r'_{\text{in}}
\end{bmatrix} \), expression (4) \( \begin{bmatrix}
    r'_{\text{out}} \\
    r''_{\text{out}}
\end{bmatrix} = \begin{bmatrix}
    1 & 0 \\
    -\frac{2}{R_2} & 1
\end{bmatrix}
\begin{bmatrix}
    r_{\text{in}} \\
    r'_{\text{in}}
\end{bmatrix} \), and expression (10) \( \begin{bmatrix}
    r'_{\text{out}} \\
    r''_{\text{out}}
\end{bmatrix} = \begin{bmatrix}
    1 & 0 \\
    -\frac{2}{R_1} & 1
\end{bmatrix}
\begin{bmatrix}
    r_{\text{in}} \\
    r'_{\text{in}}
\end{bmatrix} \) indicates that,

\[
A \text{ deflection by a thin lens of focal length } f \\
\text{is equivalent to} \\
\text{reflection from a mirror of radius of curvature } R = 2f. \tag{11}
\]

**I.B Optical resonator**

Through expressions (4) and (10) we have 2x-2 matrices that take care of the reflection of rays between two spherical mirrors of radius \( R_1 \) and \( R_2 \) respectively.
In effect, one bouncing cycle between these two mirrors, reflection at the mirror of radius \( R_2 \), translation towards the left mirror, and reflection at the mirror of radius \( R_1 \) will be described by the successive application of the corresponding 2x2 matrices,

\[
\begin{bmatrix}
  r_{s+1} \\
  r'_{s+1}
\end{bmatrix} = \begin{bmatrix}
  1 & 0 \\
  -\frac{2}{R_1} & 1
\end{bmatrix} \begin{bmatrix}
  1 & d \\
  0 & 1
\end{bmatrix} \begin{bmatrix}
  1 & 0 \\
  -\frac{2}{R_2} & 1
\end{bmatrix} \begin{bmatrix}
  1 & d \\
  0 & 1
\end{bmatrix} \begin{bmatrix}
  r_s \\
  r'_s
\end{bmatrix}
\]

(12)

What remains to be evaluated is what happens after multiple cycle’s reflections. Such an analysis, however, will be performed in an equivalent system: the ray propagation in a periodic lens waveguide. That is done in the next section (afterwards we will come back to the analysis of (12).

I.C Propagation through a periodic lens system

Fig. 8 shows the propagation of rays through a periodic lens system, which consists of lenses of focal lengths \( f_1 \) and \( f_2 \), separated by a distance \( d \).

[Notice, based on expression (11), this optical system will be equivalent to light propagation inside an optical resonator with mirrors of radii of curvature \( R_1 = 2f_1 \) and \( R_2 = 2f_2 \).]

Using expressions 1 and 2, one obtains the following result for ray propagation through a unit cell,

\[
\begin{bmatrix}
  r_{s+1} \\
  r'_{s+1}
\end{bmatrix} = \begin{bmatrix}
  1 & 0 \\
  -\frac{1}{f_1} & 1
\end{bmatrix} \begin{bmatrix}
  1 & d \\
  0 & 1
\end{bmatrix} \begin{bmatrix}
  1 & 0 \\
  -\frac{1}{f_2} & 1
\end{bmatrix} \begin{bmatrix}
  1 & d \\
  0 & 1
\end{bmatrix} \begin{bmatrix}
  r_s \\
  r'_s
\end{bmatrix}
\]

(13)
Notice that expression (13) is equivalent to expression (12). Hence, an analysis of (13) will qualify to (12) as well.

Working out expression (13) gives,

\[
\begin{bmatrix}
  r_{s+1} \\
  r'_{s+1}
\end{bmatrix}
= \begin{bmatrix}
  1 & d \\
  -\frac{1}{f_1} & -\frac{d}{f_1} + 1
\end{bmatrix}
\begin{bmatrix}
  1 & d \\
  -\frac{1}{f_2} & -\frac{d}{f_2} + 1
\end{bmatrix}
\begin{bmatrix}
  r_s \\
  r'_s
\end{bmatrix}
\]

\[
\begin{bmatrix}
  r_{s+1} \\
  r'_{s+1}
\end{bmatrix}
= \begin{bmatrix}
  1 - \frac{d}{f_2} \\
  -\frac{1}{f_1} - \frac{1}{f_2}(1 - \frac{d}{f_1})
\end{bmatrix}
\begin{bmatrix}
  d (2 - \frac{d}{f_2}) \\
  -\frac{d}{f_1} + (1 - \frac{d}{f_2})(1 - \frac{d}{f_1})
\end{bmatrix}
\begin{bmatrix}
  r_s \\
  r'_s
\end{bmatrix}
\]

\[
= \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\begin{bmatrix}
  r_s \\
  r'_s
\end{bmatrix}
\text{ propagating through a unit cell}
\]

(14)

It can be shown that,

\[
AB - CD = 1
\]

(15)

Based on (14), the propagation through \( N \) unit cells gives,

\[
\begin{bmatrix}
  r_N \\
  r'_N
\end{bmatrix}
= \begin{bmatrix} A & B \end{bmatrix}^N
\begin{bmatrix}
  r_0 \\
  r'_0
\end{bmatrix}
\text{ propagating through} \ N \text{ cells}
\]

(16)
Without proof we will accept the following Chebyshev’s identity,

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^N = \begin{bmatrix}
AU_{N-1} - U_{N-2} & CU_{N-1} \\
CU_{N-1} & AU_{N-1} - U_{N-2}
\end{bmatrix}
\]

where

\[
U_N = \frac{\sin(N+1)\theta}{\sin \theta} \quad \text{with} \quad \theta = \cos^{-1} \left( \frac{A + D}{2} \right)
\]

For a ray to remain confined by the lens system shown in fig. 8, a necessary condition is that the parameter \( \theta \) be a real number. (If \( \theta \) were a complex number the \( \sin (N+1)\theta \) function become hyperbolic, and the powers of \( N \) makes the components of the matrix in (17) to become infinite.) This implies that a necessary condition for a ray to remain confines in the lens system is,

\[
\left| \frac{A + D}{2} \right| \leq 1
\]

Taking the values of \( A \) and \( D \) from expression (14),

\[
A + D = 1 \cdot \frac{d}{f_2} - \frac{d}{f_1} + (1 - \frac{d}{f_2})(1 - \frac{d}{f_1})
\]

\[
= \frac{d}{f_2} + (1 - \frac{d}{f_1}) + (1 - \frac{d}{f_2})(1 - \frac{d}{f_1})
\]

\[
= \frac{d}{f_2} + (2 \cdot \frac{d}{f_2})(1 - \frac{d}{f_1})
\]

\[
= \frac{d}{f_2} + (2 \cdot \frac{d}{f_2})(2 \cdot \frac{d}{f_1}) - (2 \cdot \frac{d}{f_2})
\]

\[
= (2 \cdot \frac{d}{f_2})(2 \cdot \frac{d}{f_1}) - 2
\]

\[
\frac{A + D}{2} = 2(1 - \frac{d}{2f_2})(1 - \frac{d}{2f_1}) - 1
\]

The condition (18) becomes,

\[-1 \leq \frac{A + D}{2} \leq 1\]
\[-1 \leq 2(1 - \frac{d}{2f_2})(1 - \frac{d}{2f_1}) - 1 \leq 1\]

\[0 \leq 2(1 - \frac{d}{2f_2})(1 - \frac{d}{2f_1}) \leq 2\]

\[0 \leq (1 - \frac{d}{2f_2})(1 - \frac{d}{2f_1}) \leq 1\]  \hspace{1cm} \text{condition for confined propagation in the lens waveguide} (19)

**Condition for light confinement in a resonator**

Since expression (12) is equivalent to expression (13), the light confinement condition (19) that applies to a periodic lens waveguide also applies to a resonator that is constituted by two concave mirrors of radii \( R_1 \) and \( R_2 \); what we have to use is replace \( f_1 = R_1/2 \) and \( f_2 = R_2/2 \),

\[0 \leq (1 - \frac{d}{R_2})(1 - \frac{d}{R_1}) \leq 1\]  \hspace{1cm} \text{condition for confined light propagation in a resonator} (20)

For the particular case that the radii are the same, the condition becomes,

\[0 \leq d \leq 2R\]  \hspace{1cm} (21)

**II. The variational principle and ray propagation in lenslike media**

Consider the propagation of rays in an optically inhomogeneous medium. In the realm of geometrical optics, a ray travels between two points \( P_1 \) and \( P_2 \) along a trajectory such that the time taken is the least. More formally, the trajectory comes out of applying a variational principle,

\[\delta \left[ \int_{P_1}^{P_2} \frac{n}{c} ds \right] = 0\]  \hspace{1cm} (22)

where \( c \) is the speed of light

\( n \) is the index of refraction (that depends on the position)

\( ds \) is the differential path element along the trajectory.

A trajectory is expressed as,

\[\bm{r}(t) = (x(t), y(t), z(t))\]  \hspace{1cm} (23)

Since
\[ ds = v \, dt = \sqrt{x'^2 + y'^2 + z'^2} \, dt \]  

where \( x' \) indicates the derivative of \( x \) with respect to \( t \), etc., expression (22) can be written as,

\[
\delta \left[ \int_{P_i}^{P_f} n(x, y, z) \sqrt{x'^2 + y'^2 + z'^2} \, dt \right] = 0 \tag{25}
\]

Let's define,

\[ F(x, x', x', x', x', t) = n(x, y, z) \sqrt{x'^2 + y'^2 + z'^2} \tag{25} \]

The condition \( \delta \left[ \int_{P_i}^{P_f} F(x, x', x', x', x', t) \, dt \right] = 0 \) is satisfied by a trajectory \( r(t) = (x(t), y(t), z(t)) \) for which, consistent

\[
\frac{\partial F}{\partial x} = \frac{d}{dt} \frac{\partial F}{\partial x'}, \quad \frac{\partial F}{\partial y} = \frac{d}{dt} \frac{\partial F}{\partial y'}, \quad \frac{\partial F}{\partial z} = \frac{d}{dt} \frac{\partial F}{\partial z'} , \tag{26}
\]

(On purpose we have omitted the procedure to attain (26) from (24) since it is exactly the same procedure followed to obtain the Lagrange Eqs. in classical mechanics from the variational principle. See, for example Section 4.1 in La Rosa's Notes on Hamilton Variational Principle.)

From (25),

\[
\frac{\partial F}{\partial x} = \frac{\partial n}{\partial x} \sqrt{x'^2 + y'^2 + z'^2} \tag{27}
\]

\[
\frac{\partial F}{\partial x'} = n \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}} \tag{28}
\]

Multiplying each side of \( \frac{\partial F}{\partial x} = \frac{d}{dt} \frac{\partial F}{\partial x'} \) by \( \frac{dt}{ds} \) one obtains

\[
\frac{\partial F}{\partial x} \frac{dt}{ds} = ( \frac{d}{dt} \frac{\partial F}{\partial x'} ) \frac{dt}{ds} \tag{29}
\]

- Working out the right side of (29),

\[
( \frac{d}{dt} \frac{\partial F}{\partial x'} ) \frac{dt}{ds} = \frac{d}{ds} \frac{\partial F}{\partial x'} = \frac{d}{ds} \left( n \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}} \right) \tag{30}
\]

- Working out the left side of (29),

\[
\frac{\partial F}{\partial x} = \frac{\partial n}{\partial x} \sqrt{x'^2 + y'^2 + z'^2}
\]
\[ \frac{\partial F}{\partial x} \frac{dt}{ds} = \frac{\partial n}{\partial x} \sqrt{x'^2 + y'^2 + z'^2} \frac{dt}{ds} = \frac{\partial n}{\partial x} \]  

(31)

Replacing (30) and (31) in (29),

\[ \frac{\partial n}{\partial x} = \frac{d}{ds} \left( n \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}} \right) \]  

(32)

Since  \[ \frac{dx}{ds} = \frac{dx}{dt} \frac{dt}{ds} = \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}}, \]  

(32) can also be expressed as,

\[ \frac{\partial n}{\partial x} = \frac{d}{ds} \left( n \frac{dx}{ds} \right) \]  

(33)

Applying (33) for each Cartesian coordinate x, y and z, one can express the result in a more compact vectorial form,

\[ \nabla n = \frac{d}{ds} \left( n \frac{dr}{ds} \right) \text{ The ray equation} \]  

(34)

where \( r(t) = (x(t), y(t), z(t)) \)

The right side gives, essentially, an indication of the ray curvature (or concavity). The left side indicate whether such a concavity is positive or negative

For example, if \( s = z \) (i.e. propagation along the z-axis) and we were analyzing the curve \( y \) vs \( z \), then, according to (33) and (34), the curvature of the path \( y(z) \) is determined by the sign of \( dn/dy \). Accordingly, the trajectory \( y(z) \) bends towards the \( y \)-region in which \( n(y) \) increases.