2. THE SIMPLE HARMONIC MOTION

2A The periodicity of harmonic functions

Here we review of some math tools

Example-1

What is the period “T" of \( x(\theta) \)?

In other words, does the value of \( x(\theta) \) repeats periodically as \( \theta \) increases?

\[
\begin{align*}
x(\theta + T) &= x(\theta) \quad \text{or} \quad \cos(\theta + T) = \cos(\theta)
\end{align*}
\]

Answer: \( T = 2\pi \)
Example-2

- \( x(t) = A \cos(3t) \) \hspace{1cm} (A is a const)

What is the period of \( x(t) \)?

\[ x(t + T) = A \cos(3(t + T)) \]

\[ = A \cos(3t + 3T) \ldots \text{has to be equal to} \]

\[ x(t) = A \cos(3t) \]

\[ x(t + T) = x(t) \] implies

\[ \cos(3t + 3T) = \cos(3t) \]

\[ \Rightarrow 3T = 2\pi \rightarrow T = \frac{2\pi}{3} \]

- What is the period of \( x(t) = A \cos(\frac{1}{5} t) \)

\[ T = \frac{2\pi}{\frac{1}{5}} = 10\pi \]

- What is the period of \( x(t) = A \cos(100 t) \)

- What is the period of \( x(t) = A \sin(6t) \)
2.B Analytical description of the SHM

2.B1 The differential equation and analytical solution

We are looking for \( x = x(t) \)

1. \[ F = -kx \]

2. \[ m \frac{d^2x}{dt^2} + kx = 0 \]

Equation of motion

We are looking for \( x = x(t) \)

**TAY-1**

Is \( x(t) = At \) a solution? (A const)

\[
\frac{dx}{dt} = A; \quad \frac{d^2x}{dt^2} = 0; \quad m(0) + kAt = kAt \neq 0
\]

**TAY-2**

Is \( x(t) = At^2 \) a solution? (A const)

\[
\frac{dx}{dt} = 2At; \quad \frac{d^2x}{dt^2} = 2A; \quad m(2A) + kAt^2 = x
\]
Try-3  Suggestion: Integrate expression 2 above

\[\frac{d^2x}{dt^2} + \frac{k}{m}x = 0\]  \hspace{1cm} (2)

\[\int \frac{d^2x}{dt^2} + \int \frac{k}{m}x = \int 0\]

we would obtain

\[\frac{dx}{dt} + \frac{k}{m} \frac{1}{2} x^2 = \text{const}\] \hspace{1cm} (3)

Is this procedure correct?

i.e. starting from (3) do we obtain (2)?

The answer is NO!

[ Hint: The error lies in the integral of \((k/m)x\). That is, expression (3) is not correct. Indeed, taking the derivative, with respect to time, of expression (3), one does not obtain expression (2). ]

Try-4  Let's try \(x(t) = A \cos(\omega t + \phi)\) \hspace{1cm} (4)

where \(A, \omega\) and \(\phi\) are unknown constants

Let's find out under what conditions (i.e. for which particular values of \(A, \omega\) and \(\phi\)) this function satisfies the equation given in (2) above.

\[\frac{dx}{dt} = -\omega A \sin(\omega t + \phi)\]
Notice, the function given in (4) will satisfy (2) if we, conveniently, choose \( w \) to be equal to \((k/m)^{1/2}\).

So: Given the differential equation

\[
m \frac{d^2 x}{dt^2} + \delta x = 0
\]

We have found the following solution

\[
x(t) = A \cos \left( \sqrt{\frac{k}{m}} t + \phi \right)
\]

where \( A \) and \( \phi \) are arbitrary constants
How does the solution $x(t) = A \cos(\omega_0 t + \phi)$ look like in a plot $x \ vs \ t$?

- **Wave-1** $x(t) = A_1 \cos(\omega_0 t)$

  ![Wave-1 Diagram]

- **Wave-2** $x(t) = A_2 \cos(\omega_0 t)$

  ![Wave-2 Diagram]

  $T = \frac{2\pi}{\omega_0}$

  Wave-2 has the same angular frequency $\omega_0$ and the same phase as wave-1. Peaks and valleys coincide.

Example of two waves having the same frequency $\omega_0$ but different phase.

$\Delta t = \frac{\phi}{\omega_0}$ (6)
Question: What is the period of the function

\[ x(t) = A \cos(w_0 t + \phi) \]

where \( w_0 = \sqrt{\frac{k}{m}} \)

Answer: \( T = \frac{2\pi}{w_0} \)

Question: What is the frequency of the function given above?

Frequency = \# cycles/sec.

In \( T \) sec \( \rightarrow \) 1 cycle

In 1 sec \( \rightarrow \) \( f \)

\[ \Rightarrow f = \frac{1}{T} \]

\( f \): Frequency

\( T \): Period.

Answer: The function \( x(t) = A \cos(w_0 t + \phi) \) has a frequency

\[ f = \frac{w_0}{2\pi} \]
2.B2 Fitting the initial conditions into the analytical solution

Example:
Block of \( m = 680 \) grams attached to a spring of \( k=65 \) N/m.
Equilibrium position at \( x = 0 \). Consider a frictionless case.
The block is pulled 11 cm and released at \( t = 0 \) with zero velocity.

a) What are the angular frequency, frequency, and the period of oscillations?

\[
\omega_0 = \left( \frac{65}{0.68} \right)^{1/2} = 9.8 \text{ rad/s.}
\]
Hence \( f = \omega_0 / 2\pi \) and \( T = 1/f \).

b) What is the amplitude \( A \) of the oscillations?
What is the value of the phase constant \( \phi \) for this motion?

Position initial condition: At \( t=0 \) the block is at \( x= 11 \) cm.
\[
x(t) = A \cos (\omega_0 t + \phi) \\
0.11 = A \cos (\phi)
\] (7)
Notice the above information (7) is not enough to figure out A and φ.

**Velocity initial condition:** An additional relationship is obtained from the initial condition for the velocity
\[ x(t) = A \cos (\omega_0 t + \phi) \] implies \[ v(t) = -A\omega_0 \sin (\omega_0 t + \phi) \]
and in particular
\[ v(0) = -A\omega_0 \sin (\phi) \]
At \( t=0 \) the problem specifies that the block is released with velocity equal to zero velocity; hence
\[ 0 = -A\omega_0 \sin (\phi) \quad (8) \]
Since A and \( \omega_0 \) are different from zero, then \( \phi \) has to be zero.
On the other hand, using this result in (7) implies that \( A = 11 \) cm.

The general solution \( x(t) = A \cos (\omega_0 t + \phi) \) then takes the form, for this particular initial conditions,
\[ x(t) = A \cos (\omega_0 t) \]
\[ = 11 \text{ cm} \cos [ (9.8 \text{ rad/s}) t ] \quad (9) \]

**Example:**
Similar information as in the previous problem:
Block of mass = 680 grams, attached to a spring of \( k=65 \text{ N/m} \). The equilibrium position is at \( x = 0 \), and there is not friction.
But this time the initial conditions are different:
At \( t = 0 \) the particle is passing by the coordinate \( x = -8.5 \) cm with a velocity of -0.92 m/s.
Find out the expression that describes the motion of the mass.

![Diagram](image)
We know the general solution:

\[ x(t) = A \cos (\omega_o t + \phi) \]

but for this particular case \( \omega_o = 9.8 \text{ rad/s} \).

The velocity is given by

\[ v(t) = -A\omega_o \sin (\omega_o t + \phi) \]

At \( t = 0 \)

\[ x(0) = A \cos (\phi) = -0.085 \text{ m} \]
\[ v(0) = -A \omega_o \sin (\phi) = -0.92 \text{ m/s} \]

which gives, \( \omega_o \tan (\phi) = -10.8 \), or

\[ \tan (\phi) = -1.1 \]

This implies

\[ \phi = -48^\circ = 0.84 \text{ rad} \]

or

\[ 132^\circ = 2.3 \text{ rad} \]

Since \( x(t) = A \cos (\omega_o t + \phi) = A \cos (\omega_o t - 0.84) \) gives a positive value at \( t = 0 \) (contrary to the initial conditions, ) the solution should be the one with the phase \( \phi = +2.3 \text{ rad} \),

\[ x(t) = A \cos (\omega_o t + 2.3 \text{ rad}) \] (10)

The velocity will be given by

\[ v(t) = -A\omega_o \sin (\omega_o t + 2.3 \text{ rad}) \]

where \( \omega_o = 9.8 \text{ rad/s} \)

At \( t=0 \) the velocity is -0.92 m/s
\[ v(0) = -A \omega_0 \sin(2.3 \text{ rad}) \] should equal -0.92 m/s

\[-A \begin{bmatrix} 9.8 \text{ m/s/s} \\ 0.74 \end{bmatrix} = -0.92 \text{ m/s} \]

which gives

\[ A = 0.13 \text{ m} \]

Final solution

\[ x(t) = 13 \text{ cm} \cos(\omega_0 t + 2.3 \text{ rad}) \], where \( \omega_0 = 9.8 \text{ rad/s} \)

**Summary**

\[ m \frac{d^2 x}{dt^2} + kx = 0 \] \quad \text{Equation of motion}

\[ x(t) = A \cos(\sqrt{\frac{k}{m}} t + \phi) \] \quad \text{General solution}

\( A, \phi \) are arbitrary constants

\[ A \text{ and } \phi \text{ are chosen to satisfy the initial conditions } (A = 0: x = x_0, v = v_0) \]
2. B3 Amplitude and Frequency in the SHM

Questions

i) If they are released at the same time, which one passes by the equilibrium position first?

ii) Which one travels faster when passing by the equilibrium position?

iii) Which one oscillates with the higher frequency?

All these questions can be answered from their corresponding $x$ vs $time$ graphs. In effect, since both have the same mass and the same sprig constant $k$, then both will oscillate with the same frequency $\omega_o = (k/m)^{1/2}$. Accordingly, both will oscillate with the same period $T_o = 1/f_o = 2\pi/\omega_o$.

Hence,

i) Both cross the equilibrium position at the same time.

ii) At the equilibrium position the slope of trace 2 is greater than the slope of trace 1; thus, particle 2 travels faster.
both oscillate with the same frequency.

The previous example helps to illustrate that:

\[ x = A_1 \left( \omega_0 t \right) \]

\[ x = A_2 \left( \omega_0 t \right) \]

\[ A_2 > A_1 \]
2.B4 Relationship between the UNIFORM CIRCULAR MOTION and the SIMPLE HARMONIC MOTION

\[ s = A \theta \]

\[ v = \frac{ds}{dt} = A \frac{d\theta}{dt} = A \omega \]

\[ v = \text{const} \Rightarrow \omega = \text{const} \]

\[ \frac{d\theta}{dt} = \omega \Rightarrow \theta = \omega t + \theta_0 \]

\[ x = A \cos(\omega t + \theta_0) \]

P moves with uniform circular motion.

\[ v = \text{const} \]

P moves with uniform circular motion.

Its projection \( P' \) undergoes a simple harmonic motion.
2.B5  Frequency (f) and angular frequency (ω)

Example: A particle "P" moves in a circle of radius $A = 40\text{ cm}$, with a constant speed $v = 80\text{ cm/s}$.

a) What is the period $T$ of the motion?

$T$ is the time needed to complete 1 cycle (one revolution)

$$T = \frac{2\pi A}{v} \Rightarrow T = \pi \text{ sec}$$

b) What is the frequency $f$?

$f$ is the number of cycles (or revolutions) the particle does in 1 sec.

$$T = \pi \text{ sec} \quad \text{1 cycle}$$

$$1 \text{ sec} \quad f = \frac{1}{T}$$

$$f = \frac{1}{\pi} \text{ Hz}$$
c) What is the angular velocity \( \omega \) of revolutions per sec.

Each revolution sweeps \( 2\pi \) radians.

Therefore

\[
\omega = 2\pi f
\]

In this particular example \( \omega = 2\pi \text{ rad/s} \times \left( \frac{\frac{1}{2}}{\text{sec}} \right) \)

\( \omega = 2 \text{ rad/sec} \)

d) How does \( \theta \) change as a function of time?

\( \theta = \omega t + \theta_0 \)

e) Write the equation for the x-component of the position of the particle.

\[
x = A \cos \theta
\]

\[
x = A \cos(\omega t + \theta_0)
\]

When referring to SHM, \( \omega \) is called angular frequency.
2C Energy Conservation in the SHM

Approach-1

\[ x(t) = A \cos(\omega_0 t + \phi) \]

When the mass "m" is at the position \( x \),
its potential energy is
\[ U = \frac{1}{2} k x^2 \]

\[ U = \frac{1}{2} k A^2 \cos^2(\omega_0 t + \phi) \]  

**Potential Energy** (1)

At that position, the mass also has a corresponding velocity \( v = -A \omega_0 \sin(\omega_0 t + \phi) \). It means the mass "m" has a kinetic energy
\[ K = \frac{1}{2} m v^2 \]

\[ K = \frac{1}{2} m \left( -A \omega_0 \sin(\omega_0 t + \phi) \right)^2 \]

\[ = \frac{1}{2} m A^2 \omega_0^2 \sin^2(\omega_0 t + \phi) \]  

**Kinetic Energy** (2)

Since \( \omega_0^2 = k/m \)
\[ K = \frac{1}{2} k A^2 \sin^2(\omega_0 t + \phi) \]  

**Kinetic Energy** (2)
Graphically, this result can be presented as follows.

At any instant time “t” the mechanical energy has the same constant value $E_o$. 

Potential energy $U = U(t)$

Kinetic energy $K = K(t)$
The conservation of mechanical energy is inherent in the equation of motion,

\[ m \frac{d^2x}{dt^2} + kx = 0 \]

Indeed, let’s apply the following trick
multiplying each term by \( \frac{d}{dt} \)

\[ m \frac{dx}{dt} \frac{d^2x}{dt^2} + \frac{dx}{dt} kx = 0 \]

Notice, this is equal to

\[ m \frac{dt}{dt} \left[ \frac{1}{2} \left( \frac{dx}{dt} \right)^2 \right] \quad \frac{dt}{dt} \left[ \frac{1}{2} kx^2 \right] \]

So,

\[ \frac{dt}{dt} \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} kx^2 \right] = 0 \]

Therefore, this term has to be a constant

\[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} kx^2 = \text{const} \]
2. D PENDULUMS

2. D1 The simple pendulum

We would like to know the frequency of the oscillations.

We apply the following general result:

\[ \text{torque} = I \alpha \]  

Both, the \textit{torque} and the \textit{momentum of inertia} have to be evaluated with respect to the axis of rotation.
Torque produced by the tension force $T$: 0

Torque produced by the gravitational force: $-L \ (mg \ Sin\theta)$

Why do we use a negative sign here?

Momentum of inertia: $I = m \ L^2$

Therefore:

$$\textbf{Torque} = (I) \ \alpha$$

$$-L \ mg \ Sin\theta = (m \ L^2) \ \frac{d\omega}{dt}$$

$$= (m \ L^2) \ \frac{d^2\theta}{dt^2}$$

$$\Rightarrow \ \frac{d^2\theta}{dt^2} + \frac{g}{l} \ Sin\theta = 0 \quad (12)$$

This equation is very difficult to solve

Therefore, we will explore a particular case: $\theta$ very small
First, an exercise

The previous table indicates that for small $\theta$

$\sin \theta \sim \theta$

and the equation of motion becomes

$$\frac{d^2 \theta}{dt^2} + \frac{g}{l} \theta = 0$$

(13)

We immediately identify the term $g/l$ with $w_0^2$. That is,

$$w_0 = \sqrt{\frac{g}{l}}$$
This means:

\[ \theta = \Theta_0 \cos(\omega_0 t + \phi) \]

For small angular displacements \( \theta \), the pendulum will oscillate with angular frequency \( \omega_0 = \sqrt{\frac{g}{L}} \).

Notice: You can control the period of the pendulum by changing the length of the cord.

The period of oscillation will be independent of the initial displacement \( \Theta_0 \).
2. D2 The physical pendulum

We would like to know the frequency of the oscillations.
\[ \tau = I \alpha \]

To continue further, let’s address first a relationship between the momenta of inertia calculated with respect to two axes that are parallel to each other, one of them passing through the center of mass (CM).
Typically we know $I_{CM}$:  

Momentum of inertia calculated with respect to the rotation axis (that passes through the pivot point)

But usually the axis of rotation does not pass through the center of mass.

If the distance of the rotation axis to the CM is $h$, the following relationship is valid:

$$I_A = I_{CM} + Mh^2$$

**Momentum of inertia:**

- **Thin bar of length $L$**  
  $$I_{CM} = \frac{1}{12}ML^2$$ (Axis perpendicular to the bar)

- **Disk of radius $R$**  
  $$I_{CM} = \frac{1}{2}MR^2$$ (Axis perpendicular to the plane of the disk)

- **Cylinder of radius $R$**  
  $$I_{CM} = \frac{1}{2}MR^2$$ (Axis along the length of the cylinder)

- **Sphere of radius $R$**  
  $$I_{CM} = \frac{2}{5}MR^2$$ (Axis perpendicular to the bar)
Practice problem 1

Example-1

The figure shows a horizontal plank of length $L=50$ cm, and mass $M=1$ Kg, pivoted at one end. The plank is also supported by a spring at 2/3 of its length, as shown in the figure; the spring constant has a value of $k=10$ N/cm.

a) **When in equilibrium, the plank is horizontal.** Evaluate the length $\delta$ the spring is compressed in this equilibrium state.

b) Assuming that the plank undergoes small amplitude oscillations, calculate the period and frequency of those oscillations.

\[ -L \, Mg \, \sin \theta = I \, \frac{d^2 \theta}{dt^2} \]

For small $\theta$:

\[ \frac{d^2 \theta}{dt^2} + \frac{MgL}{I} \, \theta = 0 \]

We identify:

\[ \omega_0 = \sqrt{\frac{MgL}{I}} \]
Practice problem 2
The figure below shows a frontal view of a thin ring, a solid sphere and a solid cylinder which are undergoing SHM (simple harmonic motion.) All of them have the same radius $R=25$ cm, but different mass (the length of the cylinder is $L=0.5\ R$). The axis of rotation is perpendicular to this page. Evaluate the correspondent period of oscillation.

Support

$M_0$
Ring

$2\ M_0$
Sphere

$4\ M_0$
Cylinder