DAMPED SIMPLE HARMONIC MOTION

Forces acting on the mass "m"

\[ F = -kx - bv \]

\[ F = ma = m \frac{d^2x}{dt^2} \text{ implies} \]

\[ m \frac{d^2x}{dt^2} + bx + kx = 0 \]
- If \( b \) were zero, we know the solution is of the form \( x(t) = A \cos(w_0 t + \phi) \).

- From real experience, we know that the amplitude of the oscillations decreases over time. So, maybe \( A \) depends on time in the following form \( A = e^{-\alpha t} \) \( (\alpha \text{ unknown}) \).

- Also, the angular frequency of the oscillations may not be equal to \( w_0 = \sqrt{k/m} \) any more, because the damping effect may tend to slow down the motion.

- Thus, tentatively, for the equation

\[
\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0
\]

let's try a solution of the form

\[
x = A_0 e^{-\alpha t} \cos(w t + \phi)
\]

\( \alpha, w, \phi, A_0 \) \text{ UNKNOWN.}
Replacing expression 2 in expression 1, one obtains

\[
\left(-w^2 + \alpha^2 - \alpha \frac{b}{m} + \frac{k}{m}\right) \cos(\omega t + \theta) + \\
+ \left(2w\alpha - \omega \frac{k}{m}\right) \sin(\omega t + \theta) =
\]

If we required that these 2 coefficients vanish (both equal to zero) then this relationship would be valid at all times \(t\).

That is exactly what we are going to do

\[
2w\alpha - \omega \frac{k}{m} = 0 \quad \Rightarrow \quad \alpha = \frac{b}{2w} \\
-w^2 + \alpha^2 - \alpha \frac{b}{m} + \frac{k}{m} = 0 \quad \Rightarrow \quad w^2 = \frac{k}{m} - \left(\frac{b}{2w}\right)
\]
Proof: $x = e^{-at}\cos(wt + \phi)$

$$
\dot{x} = e^{-at} \left[ -w \sin(wt + \phi) \right] + (-\alpha e^{-at}) \cos(wt + \phi)
$$

$$
= -w e^{-at} \sin(wt + \phi) - \alpha e^{-at} \cos(wt + \phi)
$$

$$
\ddot{x} = -w e^{-at} w \cos(wt + \phi) + w \alpha e^{-at} \sin(wt + \phi)
+ w \alpha e^{-at} \sin(wt + \phi) + \alpha^2 e^{-at} \cos(wt + \phi)
$$

$$
\dddot{x} = (-w^2 + \alpha^2) e^{-at} \cos(wt + \phi) + 2w \alpha e^{-at} \sin(wt + \phi)
$$

$$
\dddot{x} + \frac{k}{m} \dot{x} + \frac{k}{m} x = (-w^2 + \alpha^2) e^{-at} \cos(wt + \phi) + 2w \alpha e^{-at} \sin(wt + \phi)
$$

$$
- \alpha \frac{k}{m} e^{-at} \cos(wt + \phi) - \frac{k}{m} e^{-at} \sin(wt + \phi)
$$

$$
+ \frac{k}{m} e^{-at} \cos(wt + \phi)
$$

$$
\left( -w^2 + \alpha^2 - \alpha \frac{k}{m} + \frac{k}{m} \right) \cos(wt + \phi) + \left( 2w \alpha - \frac{k}{m} \right) \sin(wt + \phi) = 0
$$

$$
\Rightarrow \alpha = \frac{1}{2m}
$$

$$
\Rightarrow \frac{k}{m} = \frac{1}{4m^2} \Rightarrow \left( \frac{k}{m} - \frac{1}{2m} \right)^2 = \frac{\frac{k}{m} - \frac{1}{4m} - \frac{1}{2m}}{2m}
$$

$$
W^2 = \frac{k}{m} - \frac{1}{2m} \Rightarrow W^2 = \frac{k}{m} - \left( \frac{1}{2m} \right)^2
$$
We have found a solution for the equation

\[ \frac{d^2 x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \omega_0^2 x = 0 \quad (\omega_0^2 = \sqrt{\frac{k}{m}}) \]

It is

\[ x = A_0 e^{-\frac{b}{m} t} \cos (\omega t + \phi) \]

where \( \omega^2 = \omega_0^2 - \left(\frac{b}{m}\right)^2 \)

\[ e^{-(b/m)t} \]

\[ \text{amplitude of the oscillations decreases with time.} \]
Summary:
For the equation
\[
\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \omega_o^2 x = 0
\]
Damped oscillator
we have found a solution of the form
\[
x(t) = A_o e^{-\alpha t} \cos(\omega t + \phi)
\]
where \(\alpha = b/2m\) and \(\omega^2 = \omega_o^2 - (b/2m)^2\)

\(A_o\) and \(\phi\) to be determined by the initial conditions.
• For small \( b \)
  \[
  \left( \leftrightarrow \text{for} \quad \frac{b}{2m} \ll w_0 \right)
  \]
  \[
  w^2 = w_0^2 - \left( \frac{b}{2m} \right)^2
  \]
  \[
  w \approx w_0.
  \]

• As \( b \) increases \( \rightarrow \) \( w \) decreases

Eventually, \( b \) increases as This happens
as to make \( w = 0 \) when \( b = b_c \)
\[
= 2m w_0
\]
"CRITICALLY DAMPED MOTION"

![Diagram showing critically damped and overdamped motion]

• If \( b > b_c \) \( \rightarrow \) system does not oscillate
  "OVERDAMPED MOTION"
What happens to the energy of a damped oscillator?

Here we will consider the simple case of small "b": \( \frac{b}{2m} < \omega_0 \)

For this case \( \omega = \omega_0 \)

\[ x \approx e^{-\frac{b}{2m} t} A_0 \cos(\omega_0 t + \phi) \]

Energy:

\[ E = \frac{1}{2} K (\text{Amplitude})^2 \]

\[ = \frac{1}{2} K [A_0 e^{-(b/2m)t}]^2 \]

\[ = \frac{1}{2} K A_0^2 \left( e^{-\frac{b}{2m} t} \right) \]

\[ f(t) = e^{-\frac{b}{2m} t} \]

\[ g(t) = e^{-4t} \]

\[ b_1 < b_2 \]

\[ b_1 > b_2 \]
For a damped oscillator

Amplitude $\rightarrow A_0 e^{-\frac{b}{2m}t}$, $A_0 = \text{const}$

Energy $\rightarrow \frac{1}{2} K A_0^2 e^{-\frac{b}{2m}t}$

**Diagram:**
- $E^* = \frac{1}{2} E_0$
- $t^* = ?$
- $t$ vs. time

**Question:** How long does it take for the mechanical energy to drop to one half its initial value? We are looking for $t^*$

At $t = 0$ $\rightarrow$ $E = \frac{1}{2} K A_0^2 = E_0$

At $t = t^*$ $\rightarrow$ $E^* = \frac{1}{2} K A_0^2 e^{-\frac{b}{2m}t^*}$

Condition $E^* = \frac{1}{2} E_0$ implies $e^{-\frac{b}{2m}t^*} = \frac{1}{2}$ $\rightarrow$ solve for $t^*$
\[ \ln e^{-\frac{b}{m}t} = \ln \frac{1}{2} \]

\[-\frac{b}{m} t^d = \ln 1 - \ln 2 \]

\[ t^d = \frac{m}{b} \ln 2 \]
The Quality factor \( Q \)
Characterization of the damped oscillations

\[ x = A_0 e^{-\frac{\xi t}{2}} \cos (\omega t + \phi_0) = x(t) \]

where \( \xi = \frac{m}{c} \)

\[ \omega^2 = \omega_0^2 - \left( \frac{b}{c m} \right)^2 \]
The diagram illustrates the behavior of a system for small values of parameter $b$.

The upper part shows a decay exponential function $e^{-t/\tau}$, where $t = 2\tau$.

The lower part depicts a wave pattern over a time period $2\tau$, with

$$T = \frac{2\pi}{\omega_0} \quad \text{and} \quad \omega_0 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \gamma = \frac{m\gamma}{b}.$$
A damped oscillator is often described by its $Q$, which is defined as

$$ Q = \omega_0 \tau \quad \text{Definition of the Quality Factor}$$

(where $\tau = \pi/\delta$)

$Q$ is related to the fractional energy loss per cycle. Let's see how:

**Amplitude**

$$ A_0 \rightarrow A_0 e^{-\frac{t}{2\tau}} $$

**Energy**

$$ E = \frac{1}{2} K (A_0 e^{-\frac{t}{2\tau}})^2 $$

$$ E = \frac{1}{2} K A_0^2 e^{-\frac{t}{\tau}} e^{\frac{3t}{2\tau}} $$


$$ \frac{dE}{dt} = -\frac{1}{2} K A_0^2 \frac{1}{2} e^{\frac{2t}{3\tau}} $$

$$ \frac{dE}{E} = \frac{dt}{\tau} \quad \text{fractional change in energy during an interval of time } dt \quad  \text{val of time } dt $$. 


When $dt = \frac{T}{2}$

\[
\frac{\lvert dE \rvert_{\text{cycle}}}{E} = \frac{T}{\tau} = \frac{2\pi}{Q}
\]

\[
T = \frac{2\pi}{Q}
\]

\[
\tau = \frac{m}{b}
\]

\[
Q = \frac{\omega_0}{\tau} = \omega_0 \frac{m}{b}
\]

Notice: The lower damping (lower $b$), the higher $Q$

The lower loss of energy per cycle in the oscillator.
Notice also:

\[ E = E_0 e^{-t/T} \]

Since \( Q = 2\pi \frac{T}{\tau} \), the Quality Factor is a measurement of the number of cycles it takes the oscillator to decrease its energy to 36%.
\[ E = E_0 e^{-t/\tau} = E_0 e^{-t \omega_0 \tau} \]

\[ \frac{|\Delta E|_{\text{cycle}}}{E} = \frac{T}{\tau} = \frac{2\pi}{Q} \]

If \( \omega_0 = 600 \) \( \rightarrow \) \( \frac{(\Delta E)_{\text{cycle}}}{E} = \frac{2\pi}{600} \approx 0.01 \) 1% loss per cycle

A piano or violin string sings for \( \approx 1 \text{ sec} \) after a pluck

\[ Q = \omega_0 \tau \]

Since the frequency \( \omega_0 \) of that string is of several hundred, THEN \( \omega_0 \approx (2\pi) 330 \text{ Hz} = 2 \times 10^3 \)

Such string must have a \( Q \) of the order \( 2 \times 10^3 \) \( \frac{(\Delta E)_{\text{cycle}}}{E} = \frac{2\pi}{Q} \approx 3 \times 10^{-3} \)

0.3% loss per cycle

\[ \tau = 1 \text{ sec} \text{ in this case} \]

Visible light

Atomic transition duration \( \tau = 10^{-8} \text{ sec} \)

So, \( Q \approx 10^4 \)