LINEAR MOMENTUM

Physical quantities that we have been using to characterize the motion of a particle

- Mass \( m \)
- Velocity \( \vec{v} \)
- Kinetic energy \( \frac{1}{2} m \vec{v}^2 \)
- Mechanical energy \( \frac{1}{2} m \vec{v}^2 + U \)

Now, we introduce a new quantity:

Linear momentum \( \vec{p} = m \vec{v} \) \( \text{(1)} \)

\( \vec{p} \) is a vector!
Simple example about collisions

What is the change in LINEAR MOMENTUM?

\[ \Delta \vec{p} = \vec{p}_f - \vec{p}_i = (0.2 \text{ Kg m/s}) - (-0.2 \text{ Kg m/s}) \]

\[ = 0.4 \text{ Kg m/s} \]

\[ \Delta \vec{p} \]
Exercise

\[ m_1 = 0.5 \text{ kg} \]
\[ v_1 = 2 \text{ m/s} \]
\[ v_2 = 2 \text{ m/s} \]

A ball of mass "m" collides with a wall. What is the change of its linear momentum?

Solution:

\[ \vec{P}_{\text{before}} = m \vec{v}_1 \]
\[ = \ldots \hat{x} + \ldots \hat{y} \]

\[ \vec{P}_{\text{after}} = m \vec{v}_2 \]
\[ = \ldots \hat{x} + \ldots \hat{y} \]

\[ \Delta \vec{P} = \hat{x} + \hat{y} \]
What is the LINEAR MOMENTUM of this macroscopic object?

Answer:
Add up the LINEAR MOMENTUM of every individual microscopic constituents of the macroscopic object
LINEAR MOMENTUM of a system composed of N particles

Next task:
To describe the motion of a system of particles

\[ \vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots + m_n \vec{v}_n \]  

symbolically

\[ \vec{P} = \sum_{i=1}^{N} m_i \vec{v}_i \]  

\[ \vec{P} \text{ is a vector!} \]

\[ \frac{d\vec{P}}{dt} = m_1 \frac{d\vec{v}_1}{dt} + \ldots + m_n \frac{d\vec{v}_n}{dt} \]

\[ = m_1 \vec{a}_1 + \ldots + m_n \vec{a}_n \]
CASE 1: No external forces acting on the system (only internal ones)

\[
\frac{d\mathbf{p}}{dt} = F_1 + F_2 + \ldots + F_n
\]

(3)

This term represents all forces acting on \( m_1 \).

Notice:

\[
\sum F_{\text{Internal forces}} = F_{12} + F_{21} + \ldots + F_{1n} + F_{n1} = 0
\]

(4)
According to 3 and 4

\[ \frac{d\vec{P}}{dt} = 0 \]

which implies

\[ \vec{P} = \text{constant} \]

when no net external forces are acting on the macroscopic object

Example

\[ \begin{align*}
M_1 &= 150 \text{ kg} \\
m_2 &= 100 \text{ kg}
\end{align*} \]

\[ \vec{P}_{\text{Before}} \quad \vec{P}_{\text{After}} \]

\[ P_{\text{Before}} > P_{\text{After}} \]

\[ P_{\text{Before}} = P_{\text{After}} \quad ? \]

\[ P_{\text{Before}} < P_{\text{After}} \]
Question

\[ M_1 = 100 \text{ Kg} \]
\[ m_2 = 167 \text{ Kg} \]

\[ \vec{P}_{\text{before}} \]
\[ \vec{P}_{\text{after}} \]

\[ \vec{P}_{\text{before}} > \vec{P}_{\text{after}} \]
\[ \vec{P}_{\text{before}} = \vec{P}_{\text{after}} \]
\[ \vec{P}_{\text{before}} < \vec{P}_{\text{after}} \]

The fact is:

No matter what, the linear momentum of an isolated system remains constant.
The linear momentum of a system of particles changes only when external forces act on that system, as we see below.

CASE 2: External forces (as well as internal forces, of course) act on the system.

\[ \frac{d\vec{P}}{dt} = \vec{F}_{\text{int}} + \vec{F}_{\text{external}} \]

According to expression 3,

\[ \vec{F}_{\text{external}} = \vec{F}_a + \vec{F}_b + \vec{F}_c \]

we know this term is zero.

\[ \frac{d\vec{P}}{dt} = \vec{F}_{\text{ext}} \]

Only the external forces count.
After the kick

only the gravitational force counts. (its the only external force in this case)

But, at which point of the ball does the gravitational force acts?

To address this question, let us introduce the concept of center of mass
The Center of Mass

\[ \vec{r}_{CM} = \frac{m_1 \vec{r}_1 + \ldots + m_N \vec{r}_N}{m_1 + m_2 + \ldots + m_N} = \frac{m_1 \vec{r}_1 + \ldots + m_N \vec{r}_N}{M} \]

This is a definition

(we do not have to prove it. Definitions are stated for a matter of convenience)
Example

Find the CM of the system

\[ x_{cm} = \frac{(12)(2) + (8)(6)}{12 + 8} \frac{\text{Kg} \cdot \text{cm}}{\text{Kg}} \]

\[ = 3.6 \text{ cm} \]

location of the center of mass
Example

Evaluate the CM

\[ m_1 = m_2 = m_3 = m_4 = 10 \text{ gr.} \]

\[ \mathbf{r}_{cm} = x_{cm} \mathbf{i} + y_{cm} \mathbf{j} \]

\[ X_{cm} = m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4 \]

\[ = 10 \text{ gr.a 1 cm} + 10 \text{ gr.} (-1 \text{ cm}) + 10 \text{ gr.} (-1 \text{ cm}) + 10 \text{ gr. a 1 cm} \]

\[ = 0 \]

\[ Y_{cm} = \text{ similarly} \]

\[ = 0 \]

\[ \Rightarrow \mathbf{r}_{cm} = 0 \text{ cm } \mathbf{i} + 0 \text{ cm } \mathbf{j} \]
Example

- **Thin square plate (of uniform mass density)**

- **Position of the center of mass:** \( \ldots \hat{i} + \ldots \hat{j} \)

Example

- **Position of the center of mass:** \( 0.5 \, \text{cm} \, \hat{i} - 0.5 \, \text{cm} \, \hat{j} \)

Example

- **Position of the center of mass:** \( \ldots \hat{i} + \ldots \hat{j} \)
These types of problems can be solved by symmetry arguments and some judicious tricks. To that effect, let’s show first a general result that states that
the center of mass of a “complicated geometry” object can obtained by
a) first finding the center of mass of sub-regions parts, and
b) then finding the final center of mass based on the knowledge of
the center of mass of those sub-regions.

Consider an arbitrary distribution of $N$ particles, as shown in the
figure below. The particles have been numbered, accordingly.
In general, the position of the center of mass (with respect to some
reference) is given by,

$$
\mathbf{R}_{CM} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + ... + m_N \mathbf{r}_N}{m_1 + m_2 + ... + m_N}
$$

Notice, we can break down the numerator into an arbitrary number
of subsets groups of particles (see also the other figure below):

$$
\mathbf{R}_{CM} = \left[ \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3}{m_1 + m_2 + m_3} \right] + \left[ \frac{m_4 \mathbf{r}_4 + m_5 \mathbf{r}_5 + m_6 \mathbf{r}_6 + m_7 \mathbf{r}_7}{m_4 + m_5 + m_6 + m_7} \right] + ...
$$

We also regroup in the denominator
\[
\vec{R}_{CM} = \frac{\left[m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 \right] + \left[m_4 \vec{r}_4 + m_5 \vec{r}_5 + m_6 \vec{r}_6 + m_7 \vec{r}_7 \right]}{m_1 + m_2 + m_3} + \frac{\left[m_4 \vec{r}_4 + m_5 \vec{r}_5 + m_6 \vec{r}_6 + m_7 \vec{r}_7 \right]}{m_4 + m_5 + m_6 + m_7} + \ldots
\]

Group-A has a mass equal to \((m_1 + m_2 + m_3)\), let’s call it \(M_A\)
Group-B has a mass equal to \((m_4 + m_5 + m_6 + m_7)\), let’s call it \(M_B\)

Notice the numerator can be re-written as follows

\[
\vec{R}_{CM} = \frac{(M_A) \left[m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 \right]}{M_A} + \frac{M_B \left[m_4 \vec{r}_4 + m_5 \vec{r}_5 + m_6 \vec{r}_6 + m_7 \vec{r}_7 \right]}{M_B} + \ldots
\]

That is, when trying to find the CM of an object whose mass is distributed in a somewhat complicated geometrical distribution, we can sub-divide the object into smaller simpler objects. Each division can be represented by its corresponding CM.
Example

\[ \mathbf{r}_{cm} = ? \]

Trick

\[ \mathbf{r}_{cm,AB} = \frac{m_A \mathbf{r}_A + m_B \mathbf{r}_B}{m_A + m_B} \]

Therefore

\[ 0 = m_A \mathbf{r}_A + m_B \mathbf{r}_B \]

or

\[ \mathbf{r}_A = -\frac{m_B}{m_A} \mathbf{r}_B \]

\[ \mathbf{r}_A = -\frac{1}{3} \left( \frac{1}{2} \mathbf{x} + \frac{1}{2} \mathbf{y} \right) = -\frac{1}{6} \mathbf{x} + \frac{1}{6} \mathbf{y} \]
Position of CM

Triangular Plate
CM: Point of intersection of the three medians
(median: from vertex to midpoint of opposite side)

Regular Polygon Plates
Circular Plate
CM: At the geometrical center of the figure

Cylinder and Sphere
CM: At the geometrical center of the figure

Pyramid and Cone
CM: On line joining vertex with center of base and at 1/4 of the length measured from the base
Motion of the Center of Mass

\[ \vec{r}_{cm} = \frac{1}{M} \left( m_1 \vec{r}_1 + \ldots + m_N \vec{r}_N \right) \]

\[ \frac{d}{dt} \vec{v}_{cm} = \frac{1}{M} \left( m_1 \vec{v}_1 + \ldots + m_N \vec{v}_N \right) \]

\[ \Rightarrow \quad \vec{P} = M \vec{V}_{cm} \]

So, it is like all the mass of the system were concentrated on \( \vec{F}_{cm} \).

This system of particles can be replaced by

\[ M = m_1 + m_2 + m_3 + \ldots \]
We need to prove this.

For motion analysis, the macroscopic ball can be replaced by a point object located at the center of mass.
LINEAR MOMENTUM \( \vec{P} \)

\[
\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots + m_N \vec{v}_N = \sum_i m_i \vec{v}_i
\]

CENTER OF MASS \( \vec{P}_{cm} \)

\[
\vec{P}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots + m_N \vec{v}_N}{m_1 + m_2 + \ldots + m_N} = \frac{1}{M} \sum_i m_i \vec{v}_i
\]

Notice:

\[
\vec{v}_{cm} = \frac{d\vec{P}_{cm}}{dt} = \frac{1}{M} \sum_i m_i \vec{v}_i = \frac{1}{M} \vec{P}
\]

This suggests that the momentum of the system is the same as it would be if all the mass were concentrated at the center of mass \( \vec{P}_{cm} \) moving with velocity \( \vec{v}_{cm} \)
Expression 3 also implies

\[
\frac{d \vec{v}_{cm}}{dt} = \frac{1}{M} \frac{d}{dt} \sum_i m_i \vec{v}_i = \frac{1}{M} \sum_i m_i \frac{d \vec{v}_i}{dt} = \frac{1}{M} \sum_i m_i \frac{d \vec{q}_i}{dt} = \vec{F}_{i,\text{int}} + \vec{F}_{i,\text{ext}}
\]

\[
= \frac{1}{M} \left\{ \sum_i \vec{F}_{i,\text{int}} + \sum_i \vec{F}_{i,\text{ext}} \right\} = 0 \quad \vec{F}_{\text{ext}}
\]

\[\vec{a}_{cm} = \frac{1}{M} \vec{F}_{\text{ext}}\]

The center of mass of a system of particles moves as if it were a particle of mass equal to the total mass \(M\) of the system and subject to the external force applied to the system, \(\vec{F}_{\text{ext}}\).
Consequences

When there are not external forces, we know $\vec{P} = \text{const}$

Therefore

Since $\vec{P} = (\sum m_i) \vec{v}_{cm}$, $\vec{v}_{cm}$ is constant

Collision between 2 bodies

They interchange momentum, but the center of mass moves at constant velocity.
→ If there are external forces, we know \[ \frac{d\vec{P}}{dt} = \vec{F}_{\text{ext}} \]

Therefore \[ \frac{d}{dt} M \vec{V}_{\text{cm}} = \vec{F}_{\text{ext}} \]

\[ M \vec{a}_{\text{cm}} = \vec{F}_{\text{ext}} \]
Assume no friction from the floor.

Before collision

\[ P_1, \quad P_2 \]

After collision

\[ P_2 \quad P_3 \]

What is the relationship between \( P_1 \) and \( P_2 \)?

\[ P_1 > P_2 \quad P_1 < P_2 \]

\[ P_1 = P_2 \]

\[ P_1 > P_3 \quad P_1 < P_3 \]

\[ P_1 = P_3 \]
Example: The ballistic pendulum
The objective is to find the speed $v$ of the bullet

A block of wood is hanging from two long cords.
A bullet is fired into the block, coming quickly to rest.
The block+bullet system then swing upwards raising a vertical
distance $h=6.3$ cm

![Diagram of ballistic pendulum with labels for $M$, $m$, $v$, and $h$]

$M = 5.4$ kg
$m = 9.5$ g
$h = 6.3$ cm

Procedure-1
Initial kinetic energy:   $(1/2) m v^2$
Initial potential energy:   0

Final kinetic energy:   0
Final potential energy:   $(m+M)gh$

The conservation of the mechanical energy implies
$(1/2) m v^2 = (m+M)gh$

From which we can solve for $v$.

What is wrong with the procedure above?
Answer:
The conservation of mechanical-energy is not valid in this case. Part of the bullet’s initial kinetic energy is dissipated as heat
Is the work / kinetic-energy theorem $\Delta K = W$ valid in this case?
Answer:
Yes

Procedure-2

Step-1  Conservation of linear momentum
We assume that the collision is very brief, such that
a) During the collision the forces on the block (gravitation and tension from the cord) are balanced.
That is, no net external force is acting on the bullet-block system. Hence, the system can be considered isolated and, therefore, its total linear momentum is conserved
b) The collision is in one dimension (just after the collision the system moves horizontally.)
Before the collision  $P_i = mv$

Just after the collision, the system moves with velocity $V$ (unknown yet)

$$ P_f = (m + M)V $$

The conservation of the linear momentum implies

$$ mv = (m + M)V $$

$$ V = \left[ \frac{m}{m + M} \right] v \quad (1) $$

Step-2  Conservation of the mechanical energy
After the collision there are not dissipative forces.

Initial kinetic energy: $(1/2) (m + M) V^2$

$$ (1/2) (m +M) \left[ \frac{m^2}{(m + M)^2} \right] v^2 $$

$$ (1/2) \left[ \frac{m^2}{m + M} \right] v^2 $$

Initial potential energy: 0
Final kinetic energy: 0
Final potential energy: \((m+M)gh\)

The conservation of the mechanical energy implies
\[
\frac{1}{2} \frac{m^2}{m+M} v^2 = (m+M)gh
\]
\[
v^2 = 2\left[ \frac{(m+M)^2}{m^2} \right]gh
\]
Substituting values \(v = 630 \text{ m/s}\)