Special case: Motion under constant acceleration

Free-fall acceleration

For large variations of the distance $R$ from the Earth, the accelerations $a_1$ and $a_2$ are very different.

But if the distance variations are of the order of meters, the acceleration does not change significantly.

Objects falling down to Earth increase their speed $9.8 \text{ m/s}$ every second:

$|a| = 9.8 \text{ m/s}^2 \equiv g$

The value of $g$ is independent of the object's mass, shape, density, etc.
Exercise. A ball is thrown up. The figure shows the magnitude of the ball's velocities as it passes through the points A, B, C, D and E. Indicate the direction of the average acceleration along the different sections of the path.

Adapting our equations to the case of free-fall motion

\[
\begin{align*}
y &= y_0 + v_0 t + \frac{1}{2} (-g) t^2 \\
v &= v_0 - g t \\
v^2 &= v_0^2 + 2 (-g) (y - y_0)
\end{align*}
\]
**Example:** A ball is thrown up with an initial velocity of 12 m/s. Evaluate the maximum height reached by the ball, as well as the time it takes to reach that peak.

**Known quantities**

At $t=0$: $y = y_0 = 0$, $v = v_o = 12$ m/s

**Unknown quantities**

Max height $y_{\text{max}}$ reached by the ball

Time $t^*$ it takes to reach the max height

With the initial conditions at $t=0$ given by $y_0 = 0$ and $v_o = 12$ m/s the equation of motion takes the form:

$$y = v_0 t - \frac{1}{2} g t^2 = 12 \frac{m}{s} \cdot t - \frac{1}{2} g t^2$$

$$v = v_0 - g t = 12 \frac{m}{s} - g t$$

$$v^2 = v_0^2 - 2 g y = (12 \frac{m}{s})^2 - 2 g y$$
In particular, when \( t = t^* \), the previous equations take the form,

\[
y_{\text{max}} = 12 \frac{m}{3} t^* - \frac{1}{2} g (t^*)^2
\]

\[
o = 12 \frac{m}{3} - g t^*
\]

\[
o = \left(12 \frac{m}{3}\right)^2 - 2g y_{\text{max}}
\]

\[\rightarrow \text{gives } \quad t^* = \frac{12 \frac{m}{3}}{g} = 1.2 \text{ sec}\]

Knowing \( t = t^* \), we can calculate \( y_{\text{max}} \)

\[
y_{\text{max}} = \frac{(12 \frac{m}{3})^2}{2g} = 7.3 \text{ m}.
\]
Exercise
Calculate the time at which the ball passes through the coordinates $y = 5\, \text{m}$. Calculate also the corresponding velocity.

\[ y = y_0 + v_0 t - \frac{1}{2} g t^2 \]
\[ y = (12\, \text{m})t - (4.9\, \text{m/s}^2) t^2 \]

When $y = 5\, \text{m}$
\[ 5 = 12t - 4.9t^2 \]

\[ 4.9t^2 - 12t + 5 = 0 \]

\[ t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
\[ t = 0.53\, \text{sec} \quad t = 1.92\, \text{sec} \]

\[ v = v_0 - gt \]
\[ = 12\, \text{m/s} - (9.8\, \text{m/s}^2) t \]

$t = 0.53\, \text{s}$ implies $v = 6.8\, \text{m/s}$
$t = 1.92\, \text{s}$ implies $v = -6.8\, \text{m/s}$
CASE: Constant velocity motion \( v = v_0 \)

\[
\begin{align*}
V &= V_0 \\
x &= x_0 + v_0 t \\
a &= 0
\end{align*}
\]

CASE: Motion under constant acceleration \( a = a_0 \)

\[
\begin{align*}
x &= x_0 + v_0 t + \frac{1}{2} a_0 t^2 \\
v &= v_0 + a_0 t \\
v^2 &= v_0^2 + 2 a_0 (x - x_0)
\end{align*}
\]
\[ v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \text{Notation} \frac{dx}{dt} \]

Instantaneous velocity
Motion under constant acceleration $a$

\[
x = x_0 + v_0 t + \frac{1}{2} a t^2
\]

\[
v = v_0 + a t
\]

\[
v^2 = v_0^2 + 2a (x - x_0)
\]

Application of the above formulas to the particular case of free fall motion

\[
y = y_0 + v_0 t - \frac{1}{2} g t^2
\]

\[
v = v_0 - g t
\]

\[
v^2 = v_0^2 - 2g (y - y_0)
\]

\[a = -9.8 \text{ m/s}^2 = -g\]
The following example is to illustrate that one can choose the origin of the system of reference at any point (whichever convenient to solve the problem). The result will be independent of that particular choice.

Example
A ball is dropped from the second floor of a building. How long will it take for the ball to reach the floor?

Option #1
We decide to place the origin of the reference at the level of the second floor.

At $t=0$: $y_0 = 0$ and $v_0 = 0$

So the EQ. of motion take the form

$$y = y_0 + v_0 t - \frac{1}{2} g t^2$$

$$= - \frac{1}{2} g t^2$$

$$v = v_0 - g t = -g t$$
Option #2
We decide to place the origin of the reference at the level of the first floor

At $t=0$: $y_0 = 7.3 \text{m}$ and $v_0 = 0$

So the EQ. of motion take the form

$$y = y_0 + v_0 t = -\frac{1}{2} g t^2$$

$$= 7.3 \text{ m} - \frac{1}{2} g t^2$$

$$v = v_0 - g t = -g t$$
When $t = t^*$, the ball reaches the floor, so $y = 0$.

$$y = ?$$

$$a = 3.3 - \frac{1}{2} g(t^*)^2$$

$$\Rightarrow t^* = \frac{2 \times 3.3 \text{m}}{9.8 \text{ m/s}^2} = 1.2 \text{ seconds}$$
The following example is to illustrate how a given problem can be solved in two systems of references, still obtaining compatible results.

**EXAMPLE**

A PH-211 student, who is at an elevator ascending at constant speed of 10 m/s (relative to the floor,) throw a ball vertically up with a initial velocity of 20 m/s (relative to the student in the elevator.) Question: What is the maximum height above the ground reached by the ball?

- First, let’s figure out the equation of motion for the particle, with respect to an observer in the floor.

\[ y(t) = y_o + v_o t - \frac{1}{2} g t^2 \]

At \( t=0 \):
- \( y_o = 28 \text{ m} \)
- \( v_o = 30 \text{ m/s} \)
- \( = 2 \text{ m} \)
- \( = 20 \text{ m/s} \)
- \( = 3 \text{ m} \)
- \( = 30 \text{ m/s} \)
· When the ball reaches its maximum height, its velocity must be zero (with respect to the floor.)

\[ v^2 = (30 \text{ m/s})^2 - 2g (y - 30 \text{ m}) \]

\[ v^2 = 0 \quad \text{when} \quad y = y_{\text{max}} \]

\[(y - 30 \text{ m}) = (30 \text{ m/s})^2 / 2g = 4.6 \times 10 \text{ m} \]

\[ y = 7.6 \times 10 \text{ m} \]
Question:
When the ball reaches $Y_{max}$ (with respect to a reference to the floor) the velocity is zero (again, with respect to the floor.) But at that instant, the PH-211 student (who is riding the elevator) will state that the velocity of the ball is NOT zero. **Is the PH-211 correct?**

The answer is affirmative

**Question: How long does it take for the ball to return and hit the floor of the elevator?**
That is, what is the total travel time of the ball since leaving the PH-211 student’s hand and hitting the elevator’s floor

\[ s(t) = y(t) \]

When the ball hits the floor of the elevator: 

\[ s(t) = y(t) \]

\[ 30 + 30t - \frac{1}{2}gt^2 = 28 + 10t \]

\[ 0 = 4.9t^2 - 20t - 2 \]
Alternative method

Since the elevator is moving at constant velocity with respect to the ground, the PH-221 student can completely ignore the reference attached to the ground and use instead his/her own reference (the one attached to the elevator).
Warning: This wouldn't be true if the elevator were accelerating.

When the ball reaches the floor:
$2 = 0$ and $t =$?
$\Rightarrow \frac{1}{2} g t^2 - 20 t - 2 = 0$
which gives $t = 4.2$ seconds. Same answer as in method.