INTERFERENCE

Interference from Thin Films

INTRODUCTION
- Reflection from a sheet of glass
- Understanding the conditions for max reflection or max transmission

THIN FILM INTERFERENCE

CASE 1: \( n_1 < n_2, \quad n_2 > n_3 \)
conditions for max reflection
condition "" transmission

Summary
Thin Film Interference

CASE 2: Film thickness \( L \ll \lambda \)

Thin Film Interference

CASE 3: \( \eta_1 < \eta_2 < \eta_3 \)
Reflection and transmission at an air-glass interface

The effects of interference can be best illustrated by observing what happens when a beam of light shines on an air-glass interface. Consider the incidence of a harmonic EM-wave:

**Calculation of the reflection** $E_r$ and transmission $E_t$ amplitudes

\[
E_r = E_i \frac{n_i - n_t}{n_i + n_t} \quad \text{for air}
\]

\[
E_r = E_i \frac{1 - 1.5}{1 + 1.5} = -0.2E_i \quad \text{for glass}
\]

\[
E_t = E_i \frac{2n_i}{n_i + n_t}
\]

\[
E_t = 0.8E_i
\]

**Calculation of the reflection** $I_r$ and transmission $I_t$ intensities

\[
I_i = <S_i> = \frac{1}{2} c \varepsilon_0 E_i^2
\]

\[
I_r = <S_r> = \frac{1}{2} c \varepsilon_0 E_r^2
\]

\[
I_t = <S_t> = \frac{1}{2} c \varepsilon_0 E_t^2
\]
\[
\frac{I_r}{I_i} = \frac{E_r^2}{E_i^2} = (-0.2)^2
\]
\[
\frac{I_r}{I_i} = 0.04
\]

\[
\frac{I_t}{I_i} = \frac{VE}{CE_0} \frac{E_t^2}{E_i^2}
\]

Since \( C = \frac{1}{\sqrt{\varepsilon_0 \mu}} \), \( V = \frac{1}{\sqrt{\varepsilon \mu}} \)

and assuming \( \mu_0 = \mu \), we have

\[
\frac{C}{V} = \sqrt{\frac{\varepsilon}{\varepsilon_0}} \quad \text{or} \quad \frac{C}{V} = (\frac{\varepsilon}{\varepsilon_0})
\]

Glass index of refraction \( n_t \)

\[
\frac{I_t}{I_i} = n_t \left( \frac{E_t}{E_i} \right)^2
\]

\[
= 1.5 (0.8)^2
\]

\[
\frac{I_t}{I_i} = 0.96
\]

\( \Rightarrow \) transmitted power flux in \text{Glass}
Incidence from AIR into GLASS

Likewise, for the case of Incidence from GLASS into AIR, one obtains,
Reflection from, and transmission through, a sheet of glass

\[ \lambda \quad \text{incident} \rightarrow \quad 100\% \]

\[ \text{AIR} \quad \leftarrow \quad \text{GLASS} \quad \rightarrow \quad \text{AIR} \quad \text{transmitted} \]

\[ \% \quad \text{reflected} \]

We would expect 4% reflection from the first interface, and another \(\approx 4\%\) reflection from the second interface. That is, a net \(\approx 8\%\) reflection.

Surprisingly, this is not what happens (in general).

Rather, depending on the sheet thickness, \(L_s\), the reflected flux energy can be up to 16% (we may wonder where the extra 8% came from) or as low as 0%!
However, we shouldn’t be surprised of this result. It can be explained in terms of waves interference:

a)

If the waves (1) and (2) happen to be out of phase by $\pi/2 \text{rad}$, then they will interfere destructively. So, there would be no reflected wave. Conservation of energy implies that 100% of the incident light will be transmitted in this case.
b) Likewise

If the waves (1) and (2) happened to be in phase and they would interfere constructively,

- The amplitude of the reflected wave would be 2E
  (assuming E is the amplitude of each individual waves (1) and (2))

- The reflected power flux would be
  \[ \langle S_r \rangle = \frac{1}{2} c \varepsilon_0 (2E)^2 \]

  \[ = 4 \cdot \frac{1}{2} \varepsilon_0 E^2 \]

  reflected power contained in just one wave, (1) or (2)

- So, the reflected power is not simple twice the power in each wave (1) or (2); rather it is 4 times that.
  This explains the 4 x 4% = 16% maximum reflection.
In short:
For a proper thickness $L_1$, it happens:

\[
\begin{align*}
\lambda & \quad \rightarrow \quad I_x, \\
& \quad \rightarrow \quad I_r \\& \quad \leftarrow \quad 16\% \\
& \quad \rightarrow \quad I_t \\& \quad \rightarrow \quad 84\%
\end{align*}
\]

For a proper thickness $L_2$, it happens:

\[
\begin{align*}
\lambda & \quad \rightarrow \quad I_x, \\
& \quad \rightarrow \quad I_r \\
& \quad \rightarrow \quad I_t \\
& \quad \rightarrow \quad 100\%
\end{align*}
\]

What remains to be done is to find out the values of $L_1$ and $L_2$. We will do it in the context of studying "interference from thin films".
Interference from thin films

Key reasoning for analyzing interference in a thin film:
Waves undergo phase shift due to
i) reflections at a interface, and
ii) while travelling through the film

CASE-1: The film has a higher index of refraction than the surrounding other two media
How to attain maximum reflection?

Given \( \lambda_0 \) (wavelength of the incident radiation when traveling in vacuum) what should be the value of the film thickness \( t \) to produce maximum reflection?
Reflection from the left-side interface:

First, we realize there will be a 180° phase shift upon reflection at the left-side interface (because $n_1 < n_2$)

At the interface: Electric fields out of phase by 180°, due to reflection

Reflection from the right-side interface:

Second, $L$ should be chosen in such a way that the wave reflected from the right-side interface locks in phase with wave (1)

Intuitively, we realize $L$ will have to be, at least, equal to $\lambda_2/4$
Notice, from the last two figures, that waves (1) and (2) will interfere constructively.

Thus, we have found a condition for maximum reflection

\[ L = \frac{\lambda_2}{4} \]

And, since \( \lambda_2 = \frac{\lambda_0}{n_2} \), we have

\[ L = \frac{\lambda_0}{4n_2} \]
Notice, \( L = \frac{\lambda_0}{4n_2} \) is not the only possible value for \( L \) that will produce max reflection.
Consider, for example \( L = 3 \frac{\lambda_2}{4} = 3 \frac{\lambda_0}{4n_2} \)

(1) and (2) interfere constructively
Generalization.

Condition for MAX REFLECTION:

\[ L = N \frac{\lambda_2}{4} \]

\[ = N \frac{\lambda_0}{4 \eta_2} \quad N = 1, 3, 5, 7, \ldots \]

This condition is valid only if \( \eta_1 < \eta_2 \) and \( \eta_2 > \eta_3 \)
CASE-1: The film has a higher index of refraction than the surrounding other two media

How to attain maximum transmission?

\[ \eta_1, \eta_2, \eta_3, \eta_1 < \eta_2, \eta_2 > \eta_3 \]

Given \( \lambda_0 \) (wavelength of the incident radiation when traveling in vacuum), what should be the value of the film thickness \( L \) to produce maximum transmission?
\( \eta_1 \) should be chosen in such a way that the double reflection produces a wave (4) that interferes constructively with wave (3).

Since \( \eta_2 > \eta_1 \), the double reflections (indicated by arrows in the figure above) do not undergo phase shifts upon reflections at the interfaces.

Intuitively, we realize \( L \) has to be at least equal to \( \lambda_2/2 \).
INCIDENT

$\eta_2$

(3)

(4)

$L = \frac{\lambda_2}{2}$

waves (3) and (4) interfere constructively
Condition for MAX TRANSMISSION

\[ L_0 = M \frac{\lambda_0}{2} \]

\[ = M \frac{\lambda_0}{2 n_2} \quad \text{where } M = 1, 2, 3, 4, \ldots \]

This condition is valid only if \( n_1 < n_2 \) and \( n_2 > n_3 \)
Summary CASE-1:

Analysis of the reflected wave

\[ n_1 < n_2 \]
\[ n_3 < n_2 \]

Interference at P (from the reflected waves)

\[ \frac{2\pi}{\lambda_2} L \]

Phase difference between A and B due only to the path difference 2L

\[ \frac{k_2 L}{2L} \]

The condition \[ 2L = m \lambda_2 \] would have given rise to constructive interference in reflection. But the condition \( n_1 < n_2 \) introduces a phase shift of \( \pi \) in the wavefront A, which causes that A and B interfere destructively at P. Therefore the above expression becomes a condition for destructive interference on reflection.

\[ 2L = m \lambda_2 \quad m = 1, 2, 3, \ldots \]

Condition for min reflection and max transmission
If we wanted max reflection the condition would be

\[ S = k_2 \cdot 2L + \pi = m \cdot 2\pi \]

\text{Phase shift due to thickness of the film}

\text{due to reflection with } m = 1, 2, 3, \ldots

\[ k_2 \cdot 2L = (m - \frac{1}{2}) \cdot 2\pi \quad m = 1, 2, 3, \ldots \]

\[ \frac{2\pi}{\lambda} \cdot 2L = (m - \frac{1}{2}) \cdot 2\pi \]

\[ \frac{2L}{\lambda} = (m - \frac{1}{2}) \quad m = 1, 2, 3, \ldots \]

\[ L = (m - \frac{1}{2}) \cdot \frac{\lambda}{2} \quad m = 1, 2, 3, \ldots \]

or

\[ L = (2m - 1) \cdot \frac{\lambda}{4} \quad m = 1, 2, 3, \ldots \]
If we want minimum reflection the condition would be

$$\delta = k_2 2L + \pi = \frac{\pi}{3\pi} \frac{\pi}{5\pi}$$

$$= (2M+1) \pi \quad M = 0, 1, 2, \ldots$$

$$k_2 2L = 2M \pi$$

$$k_2 L = M \pi \quad M = 0, 1, 2, \ldots$$

$$\frac{2\pi}{\lambda} L = M \pi$$

$$L = \frac{\lambda}{2} M \quad M = 0, 1, 2, \ldots$$
Observation

If the condition \( n_1 < n_2 \)
\( n_3 < n_2 \)

is changed for \( n_1 > n_2 \)
\( n_3 > n_2 \)

the effect of the film on the conditions for max or min reflection (or transmission) remain identical.
**Example**

\[ \lambda = 0.585\,\mu m \]

Is the light reflected by the two surfaces of the film closer to interfering fully destructively or fully constructively?

**Solution**

If destructive

\[ 2L = m \frac{\lambda}{n} \]

with \( m \): integer

\[ 2.42\,\mu m = m \frac{0.585\,\mu m}{1.33} \]

\[ 5.5 = m \quad \times \]

If constructive

\[ 2L = \frac{2m+1}{2} \frac{\lambda}{n} \]

\[ 5.5 = \frac{2m+1}{2} \quad \Rightarrow \quad m = 5 \quad \checkmark \]

Fully constructive
Example

\[
\begin{aligned}
\eta_1 &= 1 \\
\eta_3 &= 1 \\
\eta_2 &= 1.33 \\
L &= 320 \text{ nm}
\end{aligned}
\]

At what wavelength is the light reflected by the film brightest to an observer?

Solution

For the case \( \eta_1 < \eta_2 \), \( \eta_3 < \eta_2 \) the condition for constructive interference in reflection is:

\[
2L = \frac{2m+1}{2} \frac{\lambda}{\eta_2}
\]

\( m = 0, 1, 2, \ldots \)

\( \Rightarrow \) Solving for \( \lambda \)

\[
\lambda = \frac{2L \times 2\eta_2}{2m+1} = \frac{1702.4 \text{ nm}}{2m+1} \quad m = 0, 1, 2, \ldots
\]

\( m = 0 \Rightarrow \lambda_0 = 1702 \text{ nm} \) near-infrared we don't see it

\( m = 1 \Rightarrow \lambda_1 = 567 \text{ nm} \) green-yellow we do see it!

\( m = 2 \Rightarrow \lambda_2 = 340 \text{ nm} \) ultraviolet we do not see it.
Interference from thin films

CASE-2: $L << \lambda$

**Question:** What happens if the thickness $L$ of the soap film is very small compared to the wavelength $\lambda$?

\[
\begin{align*}
L &<< \lambda \\
n_1 &< n_2 \\
n_3 &< n_2
\end{align*}
\]

*Interferes destructively*  
Film will look dark regardless of the wavelength
Interference from thin films
Design of antireflective coatings

**CASE-3:** \( n_1 < n_2 < n_3 \)

How to attain minimum reflection?

What should be the value of \( L \) to avoid reflection when incident radiation of \( \lambda = 550 \text{nm} \) is used?
\[ \delta = k_2 \cdot 2L \]  
phase difference between A and B  
due only to the path difference 2L

\[ \delta = \frac{2\pi}{\lambda_2} \cdot 2L \]

- Because \( n_1 < n_2 \), ray A has undergone a phase shift of 180° upon reflection at the air-film interface

Because \( n_2 < n_3 \), ray B has undergone a phase shift of 180° upon reflection at the film-glass interface

Therefore, there is not a net phase difference between rays A and B due to reflections at
Thus, the only phase difference between rays A and B is due to the path length $2L$.

$$\delta = \frac{2\pi}{\lambda_2} \cdot 2L$$

If $2L = N \frac{\lambda_2}{2}$

where $N = 1, 3, 5, 7, \ldots$

then

$$\delta = N \pi$$

which causes destructive interference (minimum reflection)
Applying this formula to the particular problem we are solving, we obtain

\[ 2L = N \frac{550\,\text{nm}}{1.38} \]

\[ L = N \times 99.6\,\text{nm} \quad N = 1, 3, 5, 7, \ldots \]

\[ \begin{align*}
N = 1 & \Rightarrow L = 99.6\,\text{nm} \\
N = 3 & \Rightarrow L = 299\,\text{nm} \\
N = 5 & \Rightarrow L = 498\,\text{nm}
\end{align*} \]

\{ All these film thickness will work (for economic reason a company will use \( L = 99.6\,\text{nm} \) \}
Alternative method to study interference phenomena in thin films

If you already developed enough skills to solve film interference problems, you may want to skip the following pages. Below we have just an alternative way of reasoning to figure out the constructive and destructive interference in thin-films
INTERFERENCE FROM THIN FILMS

**CASE 4A**

\[ \lambda = \frac{\lambda_0}{n_1} \]

\[ \lambda_0 \text{ wavelength in vacuum} \]

Given \( n_1, n_3 \) and \( n_2 \), what should be the value of \( L \) to eliminate the REFLECTED wave?

**Solution**

Let's consider what happens to the "crest" \( A \) as it enters the film.

No phase shift in transmission

No phase shift in reflection (\( n_2 > n_3 \))
Once we have analyzed the possible phase shifts due to reflections at each interface, we proceed to analyze additional phase shifts that may come from the thickness of the film.

\[
\Delta t = \frac{2L}{v} = \frac{2L}{c} \eta_2
\]

Notice the crests "A" and "B"
Notice in the previous figure that in order to have zero reflected wave, crest A will have to interfere with crest B. 

Why?

Because when crest B reaches the interface it will undergo a phase shift of $\pi$. So, if crest A and crest B are synchronized to meet at the left interface, we will have 2 waves with a phase difference of $\pi$.

\[ T = \frac{\lambda}{c} \]

\[ m \Delta t = \Delta t \quad m = 1, 2, 3, ... \]

\[ m \frac{\lambda}{c} = \frac{2L}{c} \quad n_2 \implies \]

\[ 2L = m \frac{\lambda}{n_2} \]

Condition to cancel out the reflected wave:

Destructive interference in reflection.

$\lambda =$ wavelength in vacuum.
these 2 waves will interfere destructively

\[ n_2 > n_1 \]
\[ n_2 > n_3 \]

**Condition for Destructive Interference in Reflection**

\[ mT = \Delta t \]

\[ m = \frac{2L}{v_2} = \frac{2L}{\lambda_2 f} = \frac{2L}{\lambda_2} \ T \]

\[ m\lambda_2 = 2L \quad \text{where} \quad \lambda_2 = \frac{\lambda_0}{n_2} \]
snapshots of the incident wave for different cases of slab thickness $L$
Case:
\[ L = \frac{\lambda_n}{2} \]

\( n_2 > n_1, \quad n_2 > n_3 \)

Reflected waves interfere destructively.

Snapshot of the incident wave, reflected wave from the left side of the slab, and reflected wave from the right side of the slab.
In conclusion:

\[ \eta_1 \rightarrow \eta_2 \rightarrow \eta_3 \]

\[ \lambda_i = \frac{\lambda}{\eta_1} \]

\[ 2L_i = m \frac{\lambda_0}{\eta_2} \quad m=1,2,3,... \]

If

the reflected wave cancels out, by

incident \[ \eta_1 \rightarrow \eta_2 \rightarrow \eta_3 \]

\[ \text{means of destructive interference} \]

in reflection.

CASE 1B

**Exercise:** Following the same arguments, demonstrate that when the condition

\( 2L_i = m \frac{\lambda_0}{\eta_2} \quad m=1,2,3,... \)

above is satisfied, the transmitted wave undergoes constructive interference.
CASE 1B

It takes a time $\Delta t = \frac{2L}{v_2}$ for the crest "p" to make a round trip across the slab.

For constructive interference in transmission $L$ must be chosen in such a way that, after a round trip across the slab, crest "p" interferes with another crest - $\Theta$ or $\Theta'$ for example - that is arriving at the left side of the slab.

$mT = \Delta t$
The events $P$ and $Q$ that leave from the left side of the slab interfere constructively as they advance to the right with results in max transmission

$$m \, T = \Delta t$$

$$= \frac{2L}{v_2} = \frac{2L}{\lambda_2 f} = \frac{2L}{\lambda_2} \cdot T$$

$$\Rightarrow \boxed{m \, \lambda_2 = 2L}$$  Condition for max interference in transmission

$$n_1 \quad \begin{array}{c} \vdots \vdots \vdots \vdots \vdots \end{array} \quad n_2 \quad \begin{array}{c} \vdots \vdots \vdots \vdots \vdots \end{array} \quad n_3$$

$$n_2 > n_1, \quad n_2 > n_3$$
CASE 2A

We may also ask ourselves:

\[ n_1, n_2, n_3 \quad n_1 < n_2 \]
\[ n_3 < n_2 \]

Given \( n_1, n_2, n_3 \) and \( \lambda_0 \),
what should be the value of \( L \)
to obtain constructive interference
in reflection.

ANSWER

\[ \Delta t = \frac{(2m+1) \lambda}{2} \]
\[ \frac{2L}{V_2} = \frac{(2m+1) \lambda}{2c} \]
\[ 2L = \frac{2m+1}{2} \frac{\lambda}{n_2} \]

\( \lambda \) wavelength in vacuum
\( m = 0, 1, 2, 3, ... \)
Snapshots of the incident wave for different values of the slab thickness $L$. 
Snapshots of the incident wave, reflected wave from the left side of the slab, and reflected wave from the right side of the slab.
These 2 waves will interfere constructively.

\[ \eta_2 > \eta_1 \]
\[ \eta_2 > \eta_3 \]
\[ \tau = \text{period} = \frac{\lambda}{c} \]

Condition for constructive interference in reflection:

\[ (2m+1) \frac{T}{2} = \Delta t \]
CONCLUSION:
\[ n_1 \quad \text{incidence} \quad n_2 \quad n_3 \quad n_1 < n_2 \quad n_3 < n_2 \]

Max reflection if

\[ 2L = \frac{2m+1}{2} \frac{\lambda}{n_2} \quad m = 0, 1, 2, \ldots \]

Condition for constructive interference in reflection

CASE 2B

*Exercise:* Demonstrate that when the condition \( \text{2} \) above is satisfied, the transmitted wave cancels out (destructive interference)
Interference from Thin Films

Case 3:

Example: Sample problem 36-5; p. 879 Ch. 36

\[
\begin{array}{c|c|c}
\text{Air} & \text{Coating} & \text{Glass} \\
\hline
\eta_1 = 1 & \eta_2 = 1.38 & \eta_3 = 1.5 \\
\hline
\end{array}
\]

Incident light \( \rightarrow \)

What should be the value of \( L \) to avoid reflection when \( \lambda = 550 \text{nm} \) is used?

Notice that \( \eta_1 < \eta_2 < \eta_3 \)

Therefore, an incident wave from the film toward the glass will undergo a 180° phase shift.
These 2 waves will interfere destructively

\[ \eta_1 < \eta_2 < \eta_3 \]

T = period

\[ T = \frac{\lambda}{c} \]

Condition for destructive interference in reflection

\[ \frac{(2m+1)}{2} T = \Delta t \quad m = 0, 1, 2, ... \]

\[ \frac{(2m+1)}{2} \frac{\lambda}{c} = \frac{2L}{v_2} = \frac{2L}{c} \eta_2 \]

\[ 2L = \frac{2m+1}{2} \frac{\lambda}{\eta_2} \quad m = 0, 1, 2, 3, ... \]
Applying these formulas to the particular problem we are solving, we obtain

\[ 2L = \frac{2m+1}{2} \frac{550 \text{ nm}}{1.38} \Rightarrow L = (2m+1) 99.64 \text{ nm} \]

\[ L = l_0 = 99.64 \text{ nm} \]

\[ l_1 = 299 \text{ nm} \]

\[ l_2 = 498 \text{ nm} \]

All these film thickness will work.

(For economic reasons a company will use \( L = 99.6 \text{ nm} \))