Relationship between the electric field and magnetic field amplitudes of a harmonic EM-wave

We learned in the previous sections that $E$ and $B$ self-sustain each other (this is a consequence of the 3rd and 4th Maxwell's equation). We would expect, then, that in a harmonic EM-wave a direct relationship would exist between the electric field and magnetic field amplitudes. We show below that that is indeed the case.

Consider the following harmonic EM-wave:

\[ E = E_m \cos (kx - \omega t) \]
\[ B = B_m \cos (kx - \omega t) \]

This yields,

\[ \frac{\partial E}{\partial x} = -k E_m \sin (kx - \omega t) \]
\[ \frac{\partial B}{\partial t} = +\omega B_m \sin (kx - \omega t) \]

In the previous section we found that the 3rd ME leads to

\[ \frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \]

\[ k E_m = \omega B_m \quad \text{or} \quad B_m = \frac{E_m}{c} \]

On the other hand, it can be demonstrated (although we will not do it here) that:

$E$ and $B$ are perpendicular to one another

$E \times B$ points in the direction where the wave propagates

indicates vector-product
EM-waves transport energy

The Poynting vector

Wherever there is an electric field \( \vec{E} \), there is an energy density \( \mathcal{U}_e \) (energy per unit volume, Joules/m\(^3\)) associated to it:

\[
\mathcal{U}_e = \frac{1}{2} \varepsilon_0 E^2 \quad \text{(Joules/m}\(^3\))
\]

Wherever there is a magnetic field \( \vec{B} \), there is an energy density \( \mathcal{U}_b \) (Joules/m\(^3\)) associated to it:

\[
\mathcal{U}_b = \frac{1}{2} \mu_0 B^2
\]

In an electromagnetic wave, the electromagnetic energy density will be given by

\[
\mathcal{U} = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \mu_0 B^2
\]

we can use \( \theta = \frac{E}{c} \)

\[
= \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2} \varepsilon_0 E^2 \quad \text{where we have used} \quad c^2 = \frac{1}{\varepsilon_0 \mu_0}
\]

Using again \( B = E/c \), one obtains the following equivalent expressions:

\[
\mathcal{U} = \varepsilon_0 E^2 \quad \text{or} \quad \mathcal{U} = \frac{B^2}{\mu_0} \quad \text{or} \quad \mathcal{U} = \frac{EB}{\mu_0 c}
\]

Electromagnetic energy density (Joule/m\(^3\))
But this energy is propagating with velocity $c$

Let's consider a section of area $A$

![Diagram showing a volume with energy flow]

the amount of energy in this cubic volume is $\Delta E = \mu A \Delta x$

In an interval of time $\Delta t = \frac{\Delta x}{c}$ all the energy in that cubic volume will have passed through the section of area $A$

Energy per unit time crossing the area $A$

$$\frac{\Delta E}{\Delta t} = \frac{\mu A \Delta x}{\Delta x/c} = \mu A c$$

Energy per unit time and per unit area

$$S = \mu c = \frac{EB}{\mu_0}$$
\[
\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}
\]

**Poynting vector** indicates vector-product

\[\vec{S}\] is the flow of electromagnetic energy traveling through the space, per unit area and per unit time.

\[
\begin{bmatrix}
\frac{J}{m^2 \text{ sec}}
\end{bmatrix}
\]

\[S = \frac{1}{\mu_0} EB\]

\[S = \varepsilon_0 c E^2\]

\[S = \frac{c}{\mu_0} B^2\]

\[S:\text{ power per unit area (Watt/m}^2\)]
Average power per unit area \( \langle S \rangle \): Intensity

\[
S = \varepsilon_0 c E^2
\]

Let's assume the electric field at the position \( Q \) is of the form

\[
E = E_m \cos(\omega t)
\]

The Poynting vector takes the form:

\[
S = \varepsilon_0 c E_m^2 \cos^2(\omega t)
\]

If we take average over time:

\[
\langle S \rangle = \varepsilon_0 c E_m^2 \langle \cos^2(\omega t) \rangle
\]

\[
= \frac{1}{2}
\]

\[
I = \langle S \rangle = \frac{1}{2} \varepsilon_0 c E_m^2
\]

Sometimes, this result is given in terms of rms value of the electric field amplitude

\[
E_{\text{rms}} = \frac{E_m}{\sqrt{2}} \quad \text{(definition)}
\]

\[
l = \varepsilon_0 c E_{\text{rms}}^2
\]

Intensity

\[
\text{Power per unit area (Watt/m}^2\rangle
\]
Intensity decreases with distance from the source

CASE: Point source emitting energy all over the space

\[ P_s = \langle s_1 \rangle 4\pi r_1^2 = \langle s_2 \rangle 4\pi r_2^2 \]

That is, for a constant value of \( P_s \), the intensity varies with the distance \( r \): \( I(r) = \frac{P_s}{4\pi r^2} \)

Remember: \( I = \langle s \rangle \)
Example: Consider a light source of 250 Watts, emitting electromagnetic radiation isotropically. Assuming the source emit harmonic EM-waves, calculate the amplitude of the electric filed of the wave at 1.8 m from the source.

\[
\frac{250 \text{ W}}{4\pi R^2} = \langle s \rangle = \frac{1}{2} \varepsilon_0 c E_m^2
\]

\[
R = 1.8 \text{ m} \\
\varepsilon_0 = \\
c = \\
\Rightarrow E_m^2 = \frac{2 \times 250 \frac{\text{J}}{\text{s}}}{4\pi (1.8 \text{ m})^2 \varepsilon_0 c}
\]
Electric field \( E = \frac{\text{volts}}{m} \) 

(remember \( E \cdot d = V \))

\[
\text{Volt} = \frac{\text{work}}{\text{unit charge}} = \frac{J}{C}
\]

\[
\varepsilon_0 = 8.85 \times 10^{-12} \frac{F}{m}
\]

\[
= 8.85 \times 10^{-12} \frac{C^2}{Jm}
\]

\[
F = \frac{1}{4\pi \varepsilon_0} \frac{q^2}{r^2}
\]

\[
\text{so,} \quad \varepsilon_0 = \frac{C^2}{N \cdot m^2} = \frac{C^2}{J \cdot m}
\]

Therefore

\[
E_m^2 = \frac{2 \times 250 \frac{J}{s}}{4\pi (1.8 \text{ m})^2} \times 8.85 \times 10^{-12} \frac{C^2}{Jm} \times 3 \times 10^8 \frac{m}{s}
\]

\[
= 0.46 \times 10^4 \frac{J^2}{m^2 \cdot C^2} = 0.46 \times 10^4 \frac{V^2}{m^2}
\]

\[
E_m = 0.68 \times 10^2 \frac{V}{m}
\]

In terms of rms: \( E_{\text{rms}} = \frac{E_m}{\sqrt{2}} = 0.48 \frac{V}{m} \times 10^2 \)
Exercise: The beam diameter of a typical He-Ne laser is ~ 2 mm.
(just at the output of the laser system case)

Find the amplitudes of the electric field and magnetic field of the laser beam.

\[
I = \langle s \rangle = \frac{1}{2} \varepsilon_0 c \quad E_m^2 = \frac{\text{Power}}{\text{unit area}} = \frac{2 \text{ mW}}{\pi (1 \text{ mm})^2} = \frac{2 \times 10^{-3} \text{ J/s}}{\pi (10^{-3})^2 \text{ m}^2} = 0.64 \times 10^3 \frac{\text{J}}{\text{sm}^2}
\]

\[
\Rightarrow \quad E_m^2 = \frac{2 \times 0.64 \times 10^3 \frac{J}{\text{sm}^2}}{\varepsilon_0 c}
\]

\[
E_m = 700 \frac{\text{V}}{\text{m}}
\]

\[
E_m = 7 \times 10^2 \frac{\text{V}}{\text{m}}
\]
Note: Output laser beam is not perfectly collimated.

Solid angle $\Omega$ is measured in radians:

$$\Omega = \frac{A}{r^2}$$

where

- $A$: cross section area
- $r$: distance
Point of view 1: LIGHT IS A WAVE

- Light is a wave of electric fields and magnetic fields.
- The vector product \( \mathbf{E} \times \mathbf{B} \) gives the direction of light propagation.

\[
\langle S \rangle = \frac{1}{2} \varepsilon_0 c E_m^2
\]

The higher \( E_m \) the higher the power propagating in the wave.
However, all this nice theory could not explain the following experimental result:

- Small amplitude $E_m$
- Higher $E_m$

No matter how much you increase the incident light power (increase of $E_m$), the electron is not excited to the upper level of energy.

However

- Higher frequency light
- Photo-electric effect

How to explain it?
Point of view 2: LIGHT IS A RADIATION FIELD OF QUANTIZED ENERGY PACKETS

Light is made out of photons.

1 photon energy = $h \cdot f = \frac{hc}{\lambda}$

\[
\begin{align*}
\text{frequency of the radiation} & \\
\text{constant } \hbar & \\
\text{“Planck constant”} & \\
= 6.6 \times 10^{-34} \text{ Joule sec} & \\
= 4.14 \times 10^{15} \text{ ev sec} & \\
\end{align*}
\]

*Exercise: Find the energy of 1 photon whose wavelength is $\lambda = 500$ nm ("green light")
Point of view 3: LIGHT IS BOTH A WAVE AND A PARTICLE

\[ \text{LIGHT behaves as a WAVE} \quad \leftrightarrow \]

or any ELECTROMAGNETIC WAVE behaves as a particle

The particle nature of light was first proposed by Albert Einstein in 1905 in his explanation of the "photoelectric effect."

\[ E_{\text{photon}} = hf = 6.6 \times 10^{-34} \text{ Joule.sec.F} = 4.14 \times 10^{-15} \text{ eV.sec.F} \]

\[ = h \frac{c}{\lambda} = \frac{1240 \text{ eV nm}}{\lambda} \]

\[ \text{nm} = \text{nano-meter} = 10^{-9} \text{m} \]
Hypothesis

\[ \Delta E = \frac{hc}{\lambda} \]

Experimental observation of a discrete lines spectra

\[ \lambda_1, \lambda_2, \lambda_3, \lambda_5 \]

photographic film

Measurement of discrete values of \( \lambda \) allowed to make models of the atom
LIGHT WAVE-particle duality

The propagation of light is governed by its wave properties.

The light-matter interaction is governed by its particle-like character.

If light behaves as a particle, does it have linear momentum? angular momentum?
$E \rightarrow$ will shake the charge up and down (along the X-axis)

What about $B$?

$\vec{F}_B = q \vec{v} \times \vec{B}$

$\vec{F}_B$ acts in the direction of the propagation of light

So, the mass $m$ will gain some linear momentum along the z-axis
We are trying to argue that the incident EM-wave:

- does some work on the charge (i.e. deposit some energy \( W \)),

and, as a consequence,

- the charge feels some force \( F \)

The example above suggests:

\[
F = \frac{1}{c} \frac{dW}{dt}
\]

In any circumstance where light is being absorbed by a charge, there is a force on that charge

On the other hand \( F = dp/dt \). Here \( p \) is the momentum the charge should have gained, and which should have been delivered by the EM-wave.
\[ \frac{dp}{dt} = F = \frac{1}{c} \frac{dW}{dt} \]

Or,

\[ p = \frac{W}{c} \]

The linear momentum that light delivers is equal to the energy that is absorbed, divided by \( c \).

We already knew that light carries energy (described by the Poynting vector). Now we also understand that it carries also linear momentum. In the expression above, \( W \) is the energy put into the charge \( q \), which should have come from the EM-field (i.e. from the light).

Therefore,

\[ p = \frac{W}{c} \]

\( p \) is the linear momentum of the light
\( W \) is the energy of the light
\( c \) is the speed of light

Classical physic goes this far as far on the understanding of light.

But now we have quantum mechanics that tell us that, in many respects, light behaves as particles. We mentioned in the above sections that the energy \( W \) of a light-particle is,
\[ W = hf \]

- **\( W \)** is the energy of the light-particle
- **\( h \)** Planck’s constant
- **\( f \)** frequency of the light

This light-particle, according to the arguments given above, will carry a linear momentum equal to \( W/c \); that is,

\[ p = \frac{W}{c} = \frac{hf}{c} = \frac{h}{\lambda} \]

- **\( p \)** linear momentum of the light-particle
- **\( h \)** Planck’s constant
- **\( \lambda \)** wavelength of the light
$\hat{p}$ points in the direction of the light propagation.

$p = \frac{h}{\lambda}$
Radiation Pressure

\[ \Delta p = \frac{W}{\text{wall}} \frac{W}{c} \]

\[ \Delta p_{\text{wall}} = \frac{2 W}{c} \]
How much energy is there in that volume of cross section area $A$ and length $\Delta t$?

$$\frac{1}{A} \frac{\Delta W}{\Delta t} = I$$

$$\Delta W = I A \Delta t \quad (1)$$

$\Delta W$ will cross the surface $A$ in a time $\Delta t$.

This amount of energy will deposit on the wall.

Accordingly, the momentum transferred to the wall will be

$$\Delta p_{\text{wall}} = \frac{\Delta W}{c}$$

$$= \frac{I A \Delta t}{c}$$

So, the force on the wall is

$$F_{\text{wall}} = \frac{\Delta p_{\text{wall}}}{\Delta t} = \frac{I A}{c}$$

Therefore, the pressure on the wall is

$$P_{\text{pressure}} = \frac{F_{\text{wall}}}{A} = \frac{I}{c}$$

For total absorption.
If you had a perfect mirror on the wall

Before

After

Pressure = \(2 \frac{I}{c}\)
Example

High power laser used to compress a plasma by radiation pressure.

The laser generates pulses with peak power \(1.5 \times 10^3\) MW

\[1.5 \times 10^9\ \text{W}\]

\[1.5 \times 10^9 \frac{\text{J}}{\text{s}}\]

Find the pressure exerted on the plasma if the plasma reflects all the light beams directly back.

Answer

Momentum delivered to the plasma

\[\Delta p = \frac{\Delta W}{c}\]

Force acting on the plasma

\[
\frac{\Delta p}{\Delta t} = \frac{1}{c} \frac{\Delta W}{\Delta t} = \frac{1}{c} \frac{\Delta W}{\Delta t} = \frac{1.5 \times 10^9 \frac{\text{J}}{\text{s}}}{1.5 \times 10^9 \frac{\text{J}}{\text{s}}} = 5\ \text{N}
\]

\[F = \frac{1}{c} \times 1.5 \times 10^9 \frac{\text{J}}{\text{s}}\]

\[= 0.5 \times 10^9 \frac{\text{J}}{\text{m}} = 5\ \text{N}\]

\[p = \frac{F}{A} = \frac{5}{\left(\frac{10^{-3}}{\text{m}}\right)^2} = 5 \times 10^6 \frac{\text{N}}{\text{m}}\]

Since the light reflects back

\[p = 2 \times 5 \times 10^6 \frac{\text{N}}{\text{m}}\]
Example: What is the radiation pressure at \( R = 1.5 \text{ m} \) away from a 500 Watt light bulb? Assume the surface (on which the pressure is exerted) is perfectly absorbing.

First, let's calculate the intensity at the surface \( A \)

\[
I_A = \frac{500 \text{ Watt}}{4\pi R^2}
\]

\[
\Delta p = \frac{\Delta W}{C}
\]

\[
F = \frac{1}{C} \frac{\Delta W}{\Delta t}
\]

\[
\frac{F}{A} = \frac{1}{C} \left( \frac{1}{A} \right) \frac{\Delta W}{A \Delta t}
\]

\[
= \frac{1}{C} \frac{500 \text{ Watt}}{4\pi R^2}
\]

\[
= \frac{1}{3 \times 10^8} \times \frac{5 \times 10^2}{4\pi (1.5)^2}
\]

\[
= 0.06 \times 10^{-6} \frac{N}{m^2}
\]
MAXWELL'S RAINBOW

\[ \lambda \text{ wavelength} \]

- Diameter of human hair: 100 \( \mu \text{m} \)
- Atom size: \( A = 10^{-10} \text{m} \)
- Nucleus size: \( 10^{-15} \text{m} \)

- Radio waves
  - AM
  - FM
- Infrared
- X-rays
- Gamma rays

- Frequency:
  - .001 \( \text{kHz} \)
  - 100.3 \( \text{MHz} \)
  - \( 10^{13} \)
  - \( 10^{15} \)
  - \( 10^{19} \text{Hz} \)