Let's consider a rectangular loop (in the plane XY) of width $\Delta x$ and height $L$, and apply Faraday's law

$$\mathcal{E} = -\frac{d}{dt} \phi_m$$

$$\mathcal{E} = \int \mathbf{E} \cdot d\mathbf{l} = L \left( E_y(x+\Delta x) - E_y(x) \right)$$

$$\phi_m = B_z A = B_z \Delta x$$

$$-L \left( E_y(x+\Delta x) - E_y(x) \right) = -L \Delta x \left( \frac{d}{dt} B_z \right)$$

$$\frac{E_y(x+\Delta x) - E_y(x)}{\Delta x} = -\frac{d}{dt} B_z$$

$$\Rightarrow \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

(\text{1})
Let's consider a rectangular loop (in the plane ZX) of width $\Delta x$ and height $L$, and apply the modified Ampere's law; i.e. the one that incorporates the displacement current.

\[ \int \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \mu_0 \epsilon_0 \frac{d}{dt} \frac{\mathbf{E}}{t} \]

because there is no current in the free space.

Thus, we obtain:

\[ \frac{B_z(x + \Delta x) - B_z(x)}{\Delta x} = \frac{\partial E_y}{\partial t} \]

\[ \frac{\partial B_z}{\partial x} = -\mu_0 \epsilon_0 \frac{2E_y}{\partial t} \]
Let's combine $\textcircled{1}$ and $\textcircled{2}$.

From $\textcircled{1}$ \[ \frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial}{\partial x} \frac{\partial B_z}{\partial t} \]

From $\textcircled{2}$ \[ \frac{\partial^2 B_z}{\partial t \partial x} = -\mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \]

\[ \Rightarrow \frac{\partial^2 E_y}{\partial x^2} - \frac{\mu_0 \varepsilon_0}{\varepsilon_0} \frac{\partial^2 E_y}{\partial t^2} = 0 \]

The equation above can be re-written in the following form

\[ \frac{\partial^2 E_y}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} = 0 \]

When compared with the wave equation

\[ \frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} = 0 \]

which has solutions of the form $y = f(x \pm vt)$

we realize that:

The electric field $E_y$ constitutes a wave that propagates with velocity $v$ given by,

\[ v = \sqrt{\frac{1}{\mu_0 \varepsilon_0}} \]
That is,
$$E_y = E_y (x - v t)$$
or
$$E_y = E_y (kx - \omega t)$$ with arbitrary \(k\) and \(\omega\) provided that \(\omega/k = v\)

*Exercise:* Use equations (1) and (2) to obtain

$$\frac{\partial^2 B_z}{\partial x^2} - \frac{1}{(\mu_0 \varepsilon_0)} \frac{\partial^2 B_z}{\partial t^2} = 0$$

Verify that, in particular, the following wave

$$B_z = B_z (x - v t), \text{ where } v = \sqrt{\frac{1}{\mu_0 \varepsilon_0}}$$

satisfies the equation above.

Given the wave

$$B_z = \frac{1.6 \times 10^{-3} \text{Tesla}}{\text{const}} \cos (kx - \omega t)$$

what should be the value of \(\omega/k\) if we are told that \(B_z\) is the magnetic field of a propagating electromagnetic wave.

![Diagram of wavefront and ray](image-url)
Do we know the values of \( \varepsilon_0 \) and \( \mu_0 \)?

Yes, indeed.

\[
\varepsilon_0 = 8.85 \times 10^{-12} \, \text{F/m} \\
\mu_0 = 1.26 \times 10^{-6} \, \text{H/m}
\]

Let's calculate the velocity of an electromagnetic wave

\[
\varepsilon_0 \mu_0 = 11.1 \times 10^{-18} \, \text{s}^2/\text{m}^2
\]

\[
\frac{1}{\varepsilon_0 \mu_0} = 0.09 \times 10^{18} \, \text{m}^2/\text{s}^2
\]

\[
\sqrt{\frac{1}{\varepsilon_0 \mu_0}} = 0.3 \times 10^9 \, \text{m/s} = 300,000 \, \text{km/s} \quad \text{The speed of light.}
\]

With excitement Maxwell wrote:

"We can scarcely avoid the inference that light consists in the transverse undulation of the same medium which is the cause of electric and magnetic phenomena."

Notation \( C = 300,000 \, \text{km/s} \)
Summary:
Maxwell's equations in vacuum

\[ \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_0} \]

\[ \oint \mathbf{B} \cdot d\mathbf{A} = 0 \]

\[ \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{c^2} \oint \mathbf{B} \cdot d\mathbf{A} \]

\[ \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \oint j \cdot d\mathbf{A} + \mu_0 \varepsilon_0 \frac{d}{dt} \oint \mathbf{E} \cdot d\mathbf{A} \]

\( E \): Electric field
\( B \): Magnetic field
\( j \): Current density

\( E = E(t) \) produces \( B = B(t) \)

\[ \frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0 \]

\[ \frac{\partial^2 B}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} = 0 \]

\[ C = \sqrt{\frac{1}{\varepsilon_0\mu_0}} \]

\( C = 300,000 \) km/s

\( E = E(x, t \pm c t) \)

\( B = B(x, t \pm c t) \)