FORCED OSCILLATIONS and RESONANCE

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FORCED OSCILLATIONS and RESONANCE

You shake the system, back and forth, with angular frequency $\omega_f$.

**Forces acting on the mass $m$:**

\[-Rx - b \frac{dx}{dt} + F_0 \cos(\omega_f t)\]

**Applying Newton's 2nd Law:**

\[m \frac{d^2x}{dt^2} = \text{all the forces acting on } m\]

\[m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + Rx = F_0 \cos(\omega_f t)\]

Force communicated to the system (spring + mass...) by your hand (as shown in the figure above)
If \( F_0 \) were zero, we know the solution would be \( x = e^{-\frac{b}{2m}t} \cos(w t) \).

But such type of solution that vanishes over time is of little use for the case \( F_0 \neq 0 \).

**THE REASON IS:** The external shaker (of angular frequency \( w_f \)) is continuously pumping energy into the system. Thus, we do not expect the amplitude of the oscillator to die out.

Another point:
For very low angular frequencies \( (w_f \ll w_0) \), we expect the mass \( m \) to follow closely the external perturbation, i.e. if \( F \sim \cos(w_f t) \)
then \( x \sim \cos(w_f t) \)

For high frequencies \( (w_f > w_0) \) the mass-inertia may impede the mass \( m \) to follow closely the
external perturbation.

\[ F \sim \cos(w_f t) \]
\[ \text{then } x \sim \cos(w_f t + \phi) \]

2. "out of phase"
"F and x not in phase"

- Final point

Amplitude = \( A(w_f) \)

Amplitude may depend on \( w_f \)
\[
\frac{d^2 x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + w_0^2 x = \frac{F_0}{m} \cos(\omega_f t)
\]

**Proposed solution**

External force

\[
x(t) = A(\omega_f) \cos(\omega_f t + \phi)
\]

Amplitude

Phase difference with respect to the external force.

Replacing this function \(x\) into the differential equation, we find that in order for \(x\) to become a solution, \(A\) and \(\phi\) must follow the following relation (see attached calculations on the next page).

\[
A(\omega_f) = \frac{F_0/m}{\sqrt{(w_0^2 - \omega_f^2)^2 + (\frac{b}{m})^2 w_f^2}}
\]

Notation to indicate that the amplitude \(A\) is a function of \(\omega_f\).

\[
\tan \phi(\omega_f) = -\frac{b \omega_f}{w_0^2 - \omega_f^2}
\]

and
\[
X + \frac{b}{m} x + \frac{k}{m} x = \frac{F_0}{m} \cos \omega t
\]

\[
X = A(\omega) \cos(\omega t + \phi)
\]

\[
-\omega^2 A \cos(\omega t + \phi) - A \frac{b \omega}{m} \sin(\omega t + \phi) + \omega_0^2 A \cos(\omega t + \phi) = \frac{F_0}{m} \cos \omega t
\]

\[
\cos \omega t \cos \phi - \sin \omega t \sin \phi
\]

\[
( - \omega^2 + \omega_0^2) A \cos(\omega t + \phi) - A \frac{b \omega}{m} \sin(\omega t + \phi) = \frac{F_0}{m} \cos \omega t
\]

\[
\cos \omega t \left[ ( - \omega^2 + \omega_0^2) A \cos \phi - \frac{b \omega}{m} A \sin \phi \right] +
\]

\[
\sin \omega t \left[ ( - \omega^2 + \omega_0^2) A \sin \phi - A \frac{b \omega}{m} \cos \phi \right] = \frac{F_0}{m} \cos \omega t
\]

\[
\Rightarrow ( - \omega^2 + \omega_0^2) A \cos \phi - \frac{b \omega}{m} A \sin \phi = \frac{F_0}{m} \quad (1)
\]

\[
- ( \omega_0^2 - \omega^2) A \sin \phi = A \frac{b \omega}{m} \cos \phi \quad (2)
\]

From (3):

\[
\tan \phi = -\frac{(b/m) \omega}{\omega_0^2 - \omega^2}
\]

\[
\tan \phi = -\frac{\omega}{\omega_0^2 - \omega^2}
\]

\[
\tan \frac{\omega}{\omega_0^2 - \omega^2} = \frac{F_0}{m}
\]

Using (3) in (1):

\[
(\omega_0^2 - \omega^2) A \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{b \omega}{m}\right)^2}} - \frac{b \omega}{m} A \frac{(b/m) \omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{b \omega}{m}\right)^2}} = \frac{F_0}{m}
\]
\[ A(\omega) = \frac{F_0/m}{\sqrt{(\frac{\omega^2}{\omega_0^2} - \frac{\omega^2}{\omega^2} + \frac{(bw)^2}{m^2}}} \]

\[ -\omega \phi = -\frac{\frac{1}{m} w}{\omega_0^2 - \omega^2} \]

\[ \omega = \omega_0, \quad \phi = -\pi/2 \]

\[ \omega > \omega_0, \quad \phi = -180^\circ \]
\[ x' + \frac{b}{m} x + \frac{\kappa}{m} x = \frac{F_0}{m} \cos \omega t \]

\[ x' + \frac{b}{m} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t \]

\[ A = A_0 \cos (\omega t + \phi) \]

\[-\omega^2 \cos \omega \sin (\omega t + \phi) + \frac{b}{m} \omega \sin (\omega t + \phi) = \frac{F_0}{m} \cos \omega t \]

\[ \cos \omega t \cos \phi - \sin \omega t \sin \phi \]

\[ (-\omega^2 + \frac{\kappa}{m}) \cos \phi \cos \omega t - \frac{b}{m} \omega \sin \phi \cos \omega t \]

\[-(-\omega^2 + \frac{\kappa}{m}) \sin \phi \sin \omega t - \frac{b}{m} \omega \cos \phi \sin \omega t = \frac{F_0}{m} \cos \omega t \]

\[ A \left( -\omega^2 + \frac{\kappa}{m} \right) \cos \phi - \frac{b}{m} \omega \sin \phi = \frac{F_0}{m} \Rightarrow A \left( -\omega^2 + \omega_0^2 \right) \frac{\omega^2 - \omega_0^2}{\left( \omega - \omega_0 \right)^2 + \left( \frac{b}{m} \omega \right)^2} \frac{(b/m)^2}{\sqrt{\left( \omega - \omega_0 \right)^2 + \left( \frac{b}{m} \omega \right)^2}} \]

\[ \left( \omega^2 - \frac{\kappa}{m} \right) \sin \phi = \frac{b}{m} \omega \cos \phi \]

\[ \tan \phi = \frac{b \omega}{m \left( \omega^2 - \omega_0^2 \right)} \]

\[ \phi \quad b \omega \quad \frac{F_0}{m} \quad \frac{(\omega_0^2 - \omega^2)}{\left( \left( \omega - \omega_0 \right)^2 + \left( \frac{b}{m} \omega \right)^2 \right)^{3/2}} \]
The following triangle helps to visualize how the angle $\phi$ changes with the frequency $w_f$

\[ \tan \phi = \frac{-\frac{b}{m} w_f}{w_0^2 - w_f^2} \]

At low frequencies $w_f \approx 0$

As $w_f$ increases, OA shrinks and AB expands

At $w_f \ll w_0$

$\phi \approx -90^\circ$

At $w_f \gg w_0$

$\phi \approx -90^\circ$

At very high frequencies $w_f \gg w_0$ (OA grows faster than AB)

$\phi \rightarrow 180^\circ$
Variation of $\phi$

$\omega_f$ variable driving frequency

$\omega_0$ oscillator natural frequency (fixed value)

At low $\omega_f$

$-\frac{b}{m} \omega_f$

At higher $\omega_f$ but $\omega_f < \omega_0$

$w_f = w_0$

At $\omega_f > w_0$

$\frac{-b}{m} w_f$

$\pm 90^\circ$
\[ A_{\text{max}} = \frac{F_0 / K}{\frac{1}{\alpha} \left(1 - \frac{1}{4Q^2}\right)} \]

For \( Q \gg 1 \),
\[ A_{\text{max}} \approx Q \frac{F_0}{\kappa} \]

\[ \omega'_0 = \sqrt{\omega^2_0 - \frac{1}{2} \left(\frac{\omega}{\omega_0}\right)^2} \]

\[ = \omega_0 \sqrt{1 - \frac{1}{2Q^2}} \quad (Q = \frac{m\omega_0}{b}) \]

\[ \omega'_0 \approx \omega_0 \text{ for high } Q \]
Using a 2-D diagram to describe 1-D oscillatory motion:

The real things are:

\[ x = A_0 \cos(\omega t - \frac{\pi}{10}) \]

\[ F_x = F_0 \cos(\omega t) \]

But we will "invert" the arrows \( \vec{z} \) and \( \vec{F} \) (whose lengths are \( A_0 \) and \( F_0 \) respectively) that rotate in a 2D diagram with angular velocity \( \omega \).

In this diagram we get the real things by taking the \( x \) components of \( \vec{F} \) and \( \vec{z} \).
Using a 2-D diagram to describe the 1-D oscillatory motion:

Actual motion is along the X direction only.

\[ F_x = F_0 \cos(\omega_f t) \]
\[ x = A \cos(\omega_f t + \phi) \]

Both, A and \( \phi \) depend on \( \omega_f \).

Notice, since \( F \) lags \( \tau \) (as shown in the figure above), \( x \) is negative, \( x \) always
Be aware also of the following:

\[ B \cos(t) \]

\[ A \cos(t - \frac{\pi}{10}) \]

Notice: it appears that the \[ A \cos(t - \frac{\pi}{10}) \] curve were leading the curve \[ B \cos(t) \] curved line.

But, we will stick to our 2-D circular diagram to interpret the motion of an oscillator, where we can see that arrow \( \vec{A} \) lags by \( \frac{\pi}{10} \) arrow \( \vec{B} \).
Let's incorporate the velocity also in the 2-D diagram.

Since \( x = A \cos(\omega_f t + \phi) \) (we know \( \phi \) is negative)
then \( v_x = -A \omega_f \sin(\omega_f t + \phi) \)

which can be rewritten as

\[ v_x = A \omega_f \cos(\omega_f t + \phi + \frac{\pi}{2}) \]

We realize, \( v_x \) and \( x \) are out of phase by \( \frac{\pi}{2} \).

Such result leads us to make the following diagram:

\[ F_x = F_0 \cos(\omega_f t) \]
\[ x = A \cos(\omega_f t + \phi) \]
\[ v_x = \omega_f A \cos(\omega_f t + \phi + \frac{\pi}{2}) \]
An attractive feature of the previous diagram is that it allows to visualize the phase difference $\beta$ between $V_x$ and $F_x$. This feature is helpful because, as we know, the input power from the external force into the oscillator is equal to:

$$P_{in}(t) = F_x(t) V_x(t)$$

and a knowledge of $\beta = \beta(v)$ let us evaluate how effectively the external energy is being coupled into the oscillator.

For example, if $\beta = \frac{\pi}{2}$

$$P_{in}(t) \approx 0.$$  

But, if $\beta \approx 0$ then we expect $P_{in}$ to be maximum.

Let's be more specific and calculate $P_{in}$ explicitly.
Since $P_{in}(t)$ is also a periodic function of time, it is more useful to calculate its average value $\langle P_L(t) \rangle$ taken over a period of oscillation.

\[
P_{in}(t) = V(t) F_x(t)
\]

\[V_x(t) = A\omega_f \cos(\omega_f t + \phi + \frac{\pi}{2})\]

Notice in the previous graph that this angle is actually equal to $\beta$. Just remember $-\phi$ is negative.

\[V_x(t) = A\omega_f \cos(\omega_f t + \beta)\]

\[F_x(t) = F_0 \cos(\omega_f t)\]

\[
P_{in}(t) = A\omega_f \cos(\omega_f t + \beta) \cos(\omega_f t)
\]

\[= A\omega_f \cos^2(\omega_f t) \cos(\beta) + A\omega_f \sin(\omega_f t) \cos(\omega_f t) \sin(\beta)
\]

\[= A\omega_f \cos(\beta) \cos^2(\omega_f t) + \frac{1}{2} A\omega_f \sin(2\omega_f t) \sin(\beta)
\]
Since \( \frac{1}{T} \int_0^T \cos^2(w_f t) \, dt = \frac{1}{2} \)

\( T = \frac{2\pi}{w_f} \)

\( \frac{1}{T} \int_0^T \sin(2w_f t) \, dt = 0 \), \hspace{1cm} \text{(3 points)}

we have the following result for the average value of \( P_{in} \)

\[ \langle P_{in}(t) \rangle = \frac{1}{T} \int_0^T P(t) \, dt \]

\[ \langle P_{in}(t) \rangle = \frac{1}{2} A F_0 w_f \cos(\beta) \]

where \( A = A(w_f) \)

\[ \text{average input power from the external oscillatory force } F_0 \cos(w_f t) \]

\[ \text{going into the oscillator} \]

We know:

\[ \beta = \frac{\pi}{2} - \tan^{-1} \frac{b w_f}{w_0^2 - w_f^2} = \frac{\pi}{2} + \phi \]

\[ \frac{w_0^2 - w_f^2}{1 + \frac{b}{m} w_f} \]

This graph help us evaluate \( \cos(\beta) \)

Indeed, we can identify \( \beta \) as the complementary angle of \( 1\phi \).

\[ \frac{w_0^2 - w_f^2}{1 + \frac{b}{m} w_f} \]

So:

\[ \cos(\beta) = \frac{\frac{b}{m} w_f}{\left( (w_0^2 - w_f^2)^2 + \frac{b^2}{m^2} w_f^2 \right)^{\frac{1}{2}}} \]
Replacing the explicit values of $A = A(\omega_f)$ and $\cos(\beta)$ in the expression for $\langle P(t) \rangle$, we obtain

\[
\langle P_{in}(t) \rangle = \frac{1}{2} \frac{F_0^2}{m} \frac{b}{m} \frac{\omega_f^2}{(\omega_0^2 - \omega_f^2)^2 + \frac{b^2}{m^2} \omega_f^2} \]

\[
= \frac{1}{2} \frac{F_0^2}{m} \frac{b/m}{\left(\frac{\omega_0^2}{\omega_f^2} - 1\right) + \left(\frac{b}{m}\right)^2}
\]

\text{Average input power going into the oscillator}

$\omega_0$ is the natural frequency of the oscillator $\omega_0 = \sqrt{k/m}$

$\omega_f$ is the oscillatory angular frequency of the external force acting on the oscillator $F_0 = F_0 \cos(\omega_f t)$

\text{Notice: $\langle P_{in}(t) \rangle$ is max when $\omega_f = \omega_0$}
It is easier to understand now why $\langle P_{in}(t) \rangle$ is max when $\omega_f = \omega_0$:

Since $\phi = \phi(\omega_f)$ is an angle given by

$$\tan \phi = -\frac{\frac{k}{m} \omega_f}{\omega_0^2 - \omega_f^2}$$

and $\beta + |\phi| = \frac{\pi}{2}$,

when $\omega_0 = \omega_f \rightarrow \phi = -\frac{\pi}{2}$

and

$\beta = 0$

$\beta = 0$ means the velocity $v_x$ and the external force $F_x$ are in phase.

This is the most favorable condition to transmit energy into the oscillator.
Summary

By applying an external oscillatory force $F_x$ of angular frequency $\omega_f$

$$\Rightarrow F_x = F_0 \cos(\omega_f t)$$

the oscillator responds with an oscillatory motion described by

$$\Rightarrow x = A \cos(\omega_f t + \phi)$$

where

$$A = A(\omega_f) = \frac{F_0/m}{\left[\left(\omega_0^2 - \omega_f^2\right)^2 + \left(\frac{b}{m}\right)^2 \omega_f^2\right]^{1/2}}$$

$$\phi = \phi(\omega_f) = \arctan\left(-\frac{b/m \omega_f}{\omega_0^2 - \omega_f^2}\right)$$

and, consequently, the velocity is given by

$$\Rightarrow \mathbf{v}_x = V \cos(\omega_f t + \phi + \frac{\pi}{2}) = V \cos(\omega_f t + \beta)$$

where

$$V = \frac{(F_0/m) \omega_f}{\left[\left(\omega_0^2 - \omega_f^2\right)^2 + \left(\frac{b}{m}\right)^2 \omega_f^2\right]^{1/2}}$$
ENERGY RESONANCE

The condition at which the maximum energy is being transmitted into the oscillator by the oscillatory external force is called energy resonance condition.

From our previous analysis we know that energy resonance occurs when \( \omega_f = \omega_0 \) (which gives \( t = -\frac{\pi}{2} \))

\[
F_x = F_0 \cos(\omega_0 t)
\]

\[
x = A \cos(\omega_0 t - \frac{\pi}{2})
\]

\[
V_x = V \cos(\omega_0 t)
\]

Notice that, since the general expression for \( V(\omega_f) \) can be written as:

\[
V(\omega_f) = \frac{(F_0/m)}{\left[ \left( \frac{\omega_0^2}{\omega_f^2} - 1 \right)^2 + \left( \frac{b}{m} \right)^2 \right]^{1/2}}
\]
we realize that

\[ V \text{ is maximum when } \omega_f = \omega_0 \]

Thus, at the energy resonance condition

the kinetic energy of the oscillator

is also max.

This result makes sense, since at
the energy resonance condition the
oscillator absorbs energy very efficiently
from the external force and, therefore,
we expect the energy of the oscillator,
in particular its kinetic energy, to increase

The corresponding values of \( A \) and \( V \) at \( \omega_f = \omega_0 \)
are:

\[
A(\omega_0) = \frac{F_0}{m} \frac{1}{\frac{b}{m} \omega_0} = \frac{F_0}{mw_0} \quad Q = \frac{F_0}{K} \quad Q = \frac{F_0}{K} \quad Q = \omega_0 \frac{m}{b}
\]

\[
V(\omega_0) = \frac{F_0}{m} \frac{1}{b/m} = \frac{F_0}{mw_0} \quad Q
\]

where we have used

\[ Q = \omega_0 \frac{m}{b} \]
The previous result for $A(\omega_0)$ is interesting:

If a static force $F_0$ were applied to a spring of spring constant $K$, it would undergo a static elongation equal to $\frac{F_0}{K}$.

But, if instead of applying a constant force $F_0$, we rather apply a time-dependent force $F_0 \cos(\omega_0 t)$, where $\omega_0 = \frac{K}{m}$, then the amplitude of the spring deflections is

$$\frac{F_0}{K} Q$$

That is, the max deflection of the spring is amplified by a factor of $Q$. 

$$A = \frac{F_0}{K} Q$$
A case of energy conservation

An interesting question to ask is the following: Since at the energy resonance condition (that is, at $w_f = w_0$) energy is being pumped efficiently from the external force to the oscillator, where does all that energy go?

Certainly at $w_f = w_0$ the amplitude of oscillation grows as well as the kinetic energy. However, these two quantities reach a maximum, they do not grow forever despite the fact that the energy from the external force keeps pumping in all the time. Where does that input energy go?

The answer lies in the damping force $F = -bv_x$ acting on the oscillator. The energy of the oscillator is dissipated through damping mechanisms represented in the equation of motion by the term $-bv_x$.

To corroborate this hypothesis, let's calculate the average dissipated power produced by the damping force $F = -bv_x$. 
\[ P_{\text{dis}}(t) = F(t) V_x(t) \]
\[ = -b V_x(t) V_x(t) \]

Since \[ x = A \cos(w_f t + \Phi) \]
and
\[ V_x = -A w_f \sin(w_f t + \Phi) \]

\[ P_{\text{dis}}(t) = -b A^2 w_f^2 \sin^2(w_f t + \Phi) \]

\[ \langle P_{\text{dis}}(t) \rangle = -b A^2 w_f^2 \underbrace{\langle \sin^2(w_f t + \Phi) \rangle}_{= \frac{1}{2}} \]

\[ = -\frac{1}{2} b A^2 w_f^2 \]

Using the explicit form of \[ A = A(w_f) \]
we obtain

\[ \langle P_{\text{dis}}(t) \rangle = -\frac{1}{2} b \frac{F_0^2}{m^2} \frac{w_f^2}{(w_0^2 - w_f^2)^2 + (\frac{b}{m})^2 w_f^2} \]

Average power dissipated through the damping mechanism
- \[ b V \]

Compare this expression with \[ \langle P_{\text{tot}}(t) \rangle \] we obtained previously.
On the other hand, we would have expected that, at the energy resonance condition, the amplitude of oscillation would be also maximum. However, that is not the case.

The amplitude reaches its maximum value when the frequency \( \omega_f \) is slightly lower than \( \omega_0 \) (as we will see in the following paragraph).

Then, we talk about Amplitude Resonance rather than Energy Resonance.
Notice $\langle P_{\text{dis}}(t) \rangle = -\langle P_{\text{in}}(t) \rangle$

Average power pumped into the oscillator.

Thus, in a steady state of motion, the input power received by the oscillator from the external force $F_0 \cos(\omega_c t)$ is dissipated into the damping environment around the oscillator, which is represented by the damping force $b \dot{y}$.

This result is valid at all angular frequencies $\omega_c$, in particular at $\omega_c = \omega_0$. 
AMPLITUDE RESONANCE

For what value of $\omega_f$ is the amplitude of oscillation maximum?

$$A(\omega_f) = \frac{F_0/m}{\left[ (\omega_0^2 - \omega_f^2)^2 + \left( \frac{b}{m} \right)^2 \omega_f^2 \right]^{1/2}}$$

This expression can be re-written as

$$A(\omega_f) = \frac{F_0/m}{\left[ (\omega_0^2 - \frac{1}{2} \left( \frac{b}{m} \right)^2 - \omega_f^2)^2 + \left( \frac{b}{m} \right)^2 \omega_0^2 - \frac{1}{4} \left( \frac{b}{m} \right)^4 \right]^{1/2}}$$

Peak shifted a little bit to the left of $\omega_0$

$$\sqrt{\omega_0^2 - \frac{1}{2} \left( \frac{b}{m} \right)^2} = \omega_f$$

Amplitude is maximum when

$$\omega_f = \sqrt{\omega_0^2 - \frac{1}{2} \left( \frac{b}{m} \right)^2} = \omega_0$$
The potential energy of an oscillator is proportional to the square of the amplitude.

\[ A^2(w_f) = \frac{(F_0/m)^2}{(w_f^2 - w_0^2)^2 + \left(\frac{b}{m}\right)^2 w_0^2 - \frac{1}{4} \left(\frac{b}{m}\right)^4} \]

\( A^2 \) is max when \( w_f = w_0 \).

\[ (A_{\text{max}})^2 = \frac{(F_0/b)^2}{w_0^2 - \frac{1}{4} \left(\frac{b}{m}\right)^2} \]

\( A_{\text{max}}^2 \) is given by

\[ (A_{\text{max}})^2 \propto \left(\frac{F}{m w_0^2}\right)^2 Q^2 \]

where \( Q = \frac{w_0}{b} \).
At which frequency $A^2 = \left(\frac{1}{2}\right)(A_{\text{max}})^2$?

This happens when $\omega_f = \omega_{1/2}$ such that

$$(\omega_{1/2}^2 - \omega_0^2)^2 = \left(\frac{b}{m}\right)^2 \omega_0^2 - \frac{1}{4} \left(\frac{b}{m}\right)^4$$

To simplify things, let's consider the case $a \gg 1$ ($\iff \omega_0 \gg b/m$) where $\omega_0 \approx \omega_0$

$$(\omega_{1/2}^2 - \omega_0^2)^2 \approx \left(\frac{b}{m}\right)^2 \left[\omega_0^2 - \frac{1}{2} \left(\frac{b}{m}\right)^2\right]$$

$\approx \omega_0^2$

$$(\omega_{1/2}^2 - \omega_0^2)^2 \approx \left(\frac{b}{m}\right)^2 \omega_0^2$$

$$(\omega_{1/2} - \omega_0)(\omega_{1/2} + \omega_0) \approx \frac{b}{m} \omega_0$$

$\approx 2 \omega_0$

$\omega_{1/2} \approx \omega_0 + \frac{b}{2m}$

frequencies at which $A^2$ is half $A_{\text{max}}^2$
when $Q \gg 1$

$(A_{\text{max}})^2 \approx \left( \frac{F}{b \omega_0} \right)^2 = \left( \frac{F}{m \omega_0^2} \right)^2 Q$

$= \left( \frac{F}{k} \right)^2 Q^2$

$(\frac{1}{2})(A_{\text{max}})^2$

$\Delta w = \frac{b}{m}$

$= \frac{\omega_0}{Q}$

$Q = \omega_0 \tau$

$= \frac{\omega_0 m}{b}$

$= \frac{\omega_0}{\Delta w}$ measured by experiment

So,

$Q = \frac{\omega_0}{\Delta w}$

Experimental measurement of $Q$