REFRACTION at SPHERICAL SURFACES
Ray tracing under the Snell’s law
The paraxial approximation

Aspherical Surfaces
- Are difficult to manufacture with great accuracy
- Images of finite size are not free from aberrations (the larger the object the less precise is its image)

Aberrations:
When an optical system can not produce a one-to-one relationship between the OBJECT and the IMAGE (as required for perfect imaging of all object points) one speaks of system aberrations.

As it turns out, different applications may require different degree of precision.

That is, some (if not the great majority of) optical systems, although compromising the level of “perfect imaging,” may tolerate some degree of aberrations.

Principally, if the image detection systems (cameral film, human eye, ..., etc) do not have fine resolution, then a perfect image quality produced by a sophisticated optical imaging system would be wasted.

There is, then, room for relaxing the requirement of perfect imaging. This springs an interest for trying simpler surfaces (instead of the aspherical ones) for imaging applications. Due to its ease in fabricating them, spherical surfaces are good candidates.
Objective: The objective in Lecture-13 is to familiarize with the use of spherical surfaces as imaging elements.

Method: We will use the Snell’s law to directly evaluate the refraction of rays at the spherical surfaces. (Although we will not be invoking explicitly the least-time principle in lecture-13, the latter will be used in the next lecture-14 to evaluate the “imperfection” of “imaging through spherical surfaces” compared to “imaging through aspherical surfaces”.)

When using spherical imaging surfaces, it will become evident that unavoidable aberrations will result since not all the rays leaving the object point and reaching the surface will refract to the image point; unless the object point is very close to the optical axis. Hence, only object points located near the optical axis will be considered. This will constitute the so called paraxial or Gaussian approximation.

Being aware that spherical surfaces will produce aberrations, we would like also to quantify the degree of aberrations they produced (compared to an aspherical surface). Such quantification of the aberration will be postponed for the following lecture-14.
Imaging with spherical lenses

\[ n_0 \sin \theta_0 = n_i \sin \theta_i \]

We seek for a relationship between \( s_0 \) and \( s_i \) that depends only on the radius of curvature \( R \) and the indices of refraction \( n_0 \) and \( n_i \).

Such a simple relationship may not always be possible to obtain, unless we restrict the points on the spherical surface available for imaging. Thus, arbitrarily, let's restrict our analysis and consider points like "A" (shown in the figure) located very close to the optical axis.

In such a case, ALL the intervening angles in the figure are small, which invites to make the following approximations.
\[
\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots
\]
\[
\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots
\]

First-order or Gaussian optics \[\rightarrow\]

\[
\sin \theta = \theta \quad \text{"Paraxial approximation"}
\]
\[
\cos \theta = 1
\]

In this approx, \[\theta = \tan \theta\]
Before continuing with the effort to obtain a formula that allow us to locate the position of the image point in terms of the location of the object point, let's describe some general terminologies.

**Convention**

If the image is at the right of \( V \rightarrow S_i \), it is positive.

Real image: the image can be viewed or intercepted on a screen.

Sometimes \( S_i \) may turn out to be negative. It means that the "image" is located at the left of \( V \).

\[ S_o > 0 \quad S_i < 0 \]
When the **OBJECT** is at the **LEFT** of \( V \rightarrow \)

\( \rightarrow S_0 \) is considered to be **positive**

**REAL OBJECT**

Sometimes \( S_0 \) may turn out to be **negative**

It means that the "object" is located at the at the right side of \( V \).

We call it "virtual object"

**How could it happen that an object is "virtual"?**

*The graph below outline an answer*
Virtual objects result when the incident rays come from another imaging interface.

\[ S_i \quad S_o \]
\[ (+) \quad (-) \]

Another example: In the figure below the first spherical surface receives a real object. However, the second surface receives incident rays that (appear) to correspond to a virtual object located at \( P \).
Convention

Since the center of curvature $C$ is located at the right-side of $V$, the value of $R$ is considered positive.

$R = |R|$

$C$ located at the left-side of $V$, the value of $R$ is.

In this case, the value of $R$ will be considered negative.

$R = -|R|$
Imaging with a spherical convex \((R>0)\) lens

\[
\eta_0 \sin \theta_o = \eta_x \sin \theta_i \quad \text{Snell's law}
\]

In the paraxial approximation: \(\eta_0 \theta_o = \eta_x \theta_i\)

In the figure: \(\theta_o = \gamma + \beta, \quad \theta_i = \beta - \chi\)

In the paraxial approximation we are allowed to take only those points A very close to the optical axis. In such cases: \(\overline{VH} \approx 0\)

\[
\theta_o = \frac{h}{s_o} + \frac{h}{R}, \quad \theta_i = \frac{h}{R} - \frac{h}{s_i}
\]

\[
\eta_0 \theta_o = \eta_x \theta_i \implies \frac{\eta_0}{s_o} + \frac{\eta_x}{s_i} = \frac{\eta_x - \eta_0}{R}
\]

Expression independent of \(h\)
The Focal Point

Identifying

object at infinity $s_o = \infty$

image at $F$ $s_i = F$

therefore

$$\frac{n_o}{\infty} + \frac{n_i}{F} = \frac{n_i - n_o}{R} \Rightarrow \boxed{F = \frac{n_i}{n_i - n_o} R}$$

Notice $F > R$
Notice: \[ f = \frac{n_k}{n_k - n_0} \cdot R \] The higher \( n_k \), the smaller \( f \).

\[ \eta_k < \eta_k' \]

We'll see later on that lenses with shorter focal length \( f \) provide finer resolution. (Better capability to see smaller things)

The graphs above are also indicative that a lens with smaller \( f \) will have a light-gathering power.
Microscope objective lens

\[ \eta_{\text{oil}} \propto \eta_j \]
\[ \alpha_0 > \alpha_a \]

Oil-immersion lens

Increased light-gathering power

A measurement of the light-gathering capability is the numerical aperture NA

\[ NA = \eta \sin \alpha \]
Notice: Using the expression (2) for \( f \), expression (1) can be re-written as

\[
\frac{n_0}{s_0} + \frac{n_r}{s_i} = \frac{n_2}{f}
\]

RAY TRACING to find the images

Rays and will suffice to determine the image point \( B' \).
Once \( B' \) is determined, the trace of any other
Ray becomes straightforward.

An "aperture stop" is used to control the brightness of the image.

LATERAL MAGNIFICATION

\[
m = \frac{h_i}{h_o}
\]
In $\triangle THF$ and $\triangle FA'B'$

$$\frac{TH}{HF} = \frac{A'B'}{FA'}$$

$$\frac{TH}{HF} \approx f \Rightarrow \frac{h_o}{f} = \frac{|h_i|}{s_i - f} \quad \text{geometric relationship}$$

$$\frac{-h_i}{h_o} = \frac{s_i - f}{f}$$

OPTICS:
Image (see previous figure) is inverted. So, we consider $h_i$ negative.

Also, since $\frac{n_o}{s_o} + \frac{n_l}{s_o} = \frac{n_l}{f}$

$$n_o \frac{s_i}{s_o} + n_l = n_l \frac{s_i}{f}$$

$$\frac{n_o}{n_l} \frac{s_i}{s_o} + 1 = \frac{s_i}{f}$$

$$\frac{n_o}{n_l} \frac{s_i}{s_o} = \frac{s_i - f}{f}$$

$$\frac{-h_i}{h_o} = \frac{n_o}{n_l} \frac{s_i}{s_o}$$

$$m = \frac{h_i}{h_o} = -\frac{n_o}{n_l} \frac{s_i}{s_o}$$
From page-15 to page-23 there is no missing material (those pages has been omitted on purpose.)
Example: A real object is positioned in air, 30 cm from a convex spherical surface of radius \( R = 5 \text{ cm} \). To the right of the interface, the refractive index is \( \eta_A = 1.33 \).

a) Construct the representative rays and find the image.

To that effect, let's calculate first the focal point:

\[
f = \frac{R}{\eta_A - \eta_0} = \frac{1}{1.33} \cdot \frac{1.33 - 1}{5 \text{ cm}} = 0.20 \text{ cm}
\]

b) Find analytically the image distance \( s_i \) and the magnification.

\[
\frac{\eta_A}{s_o} + \frac{\eta_0}{s_i} = \frac{\eta_A - \eta_0}{R}
\]

\( s_o = 30 \text{ cm} \)

\( \eta_0 = 1 \)

\( \eta_A = 1.33 \)

\( R = 5 \text{ cm} \)

\[
\frac{1.33}{30 \text{ cm}} + \frac{1}{s_i} = \frac{1.33 - 1}{5 \text{ cm}}
\]

\[
\Rightarrow \quad \frac{1.33}{s_i} = \frac{0.33}{5 \text{ cm}} - \frac{1}{30 \text{ cm}} = 0.0083 \Rightarrow s_i = 40.74 \text{ cm}
\]

Image location.

Magnification:

\[
\frac{h_i}{h_0} = \frac{\eta_A}{\eta_0} \cdot \frac{s_i}{s_o} = \frac{1}{1.33} \cdot \frac{40.74 \text{ cm}}{30 \text{ cm}} \approx 1 \]

\[
\frac{h_i}{h_0} \approx -1
\]

Magnification.
Example: Imaging through a thick lens.

An object is placed 30 cm away in front of a solid sphere (n = 1.33) of radius 5 cm. Find the location of the final image and its magnification.
Imaging through surface (1):

\[ S_{o1} = 30 \text{ cm} \]

Convex \( \Rightarrow R = +5 \text{ cm} \)

\[
\frac{n_{\text{air}}}{S_{o1}} + \frac{n_e}{S_{i1}} = \frac{n_e - n_{\text{air}}}{R}
\]

\[
\frac{1}{30 \text{ cm}} + \frac{1.3}{S_{i1}} = \frac{1.33 - 1}{5 \text{ cm}}
\]

\( \Rightarrow S_{i1} = 40.7 \text{ cm} \quad \text{Image location} \)

- \( S_{i1} > 0 \) implies image is real

- Magnification: \( m_1 = -\frac{n_{\text{air}}}{n_e} \frac{S_{i1}}{S_{o1}} \)

\[
= -\frac{1}{1.33} \frac{40.7 \text{ cm}}{30 \text{ cm}} = -1
\]

- \( m_1 < 0 \) implies the image is inverted

Imaging through surface (2):

Notice in the previous figure that we need to measure the object and image with respect to point \( V_2 \).

\( S_{o2} \): The real image formed by the surface (1) becomes the object for the surface (2). Since this item is located at the right side of \( V_2 \), \( S_{o2} \) is a negative value.

\[
|S_{o2}| = S_{i1} - 10 \text{ cm} = 40.7 \text{ cm} - 10 \text{ cm} = 30.7 \text{ cm}. \quad S_{o2} = -30.7 \text{ cm} \]
\[ s_{02} = -30.7 \text{ cm} \]

\[ \text{concave} \Rightarrow R = -5 \text{ cm} \]

\[ \frac{n_R}{s_{02}} + \frac{n_{\text{air}}}{s_{i2}} = \frac{n_{\text{air}} - n_R}{R} \]

It is worth noticing that we use \( n_R \) here despite the fact that we know the position of the object ("virtual" in this case) is located in the air medium.

\[ \frac{1.33}{-30.7 \text{ cm}} + \frac{1}{s_{i2}} = \frac{1 - 1.33}{-5 \text{ cm}} \]

\[ \Rightarrow s_{i2} = 9.1 \text{ cm} \]

Location of the final image with respect to \( V_2 \).

Magnification \( m_2 = -\frac{n_R}{n_{\text{air}}} \frac{s_{i2}}{s_{02}} \)

\( \text{(produced by surface 2 only)} \)

\[ = -\frac{1.33}{1} \frac{9.1 \text{ cm}}{-30.7} = 0.4 \]

Overall magnification: \( m = m_1 \cdot m_2 \)

\[ = (-1)(0.4) = -0.4 \]

\( m < 0 \) implies the final image is inverted.
Particular case: \( R = \infty \)

\[
\frac{\eta_{\text{water}}}{S_o} + \frac{\eta_{\text{air}}}{S_i} = 0
\]

\[
\Rightarrow \\
S_i = -\frac{S_o}{\eta_{\text{water}}}
\]

\[
\begin{align*}
\eta_{\text{water}} &= 1.33 \\
S_i &= -\frac{3}{4} S_o
\end{align*}
\]

When we look at the bottom of a swimming pool from above, it does not look as deep as it really is by a factor of \( 3/4 \).