Helpful approximations

We provide here a couple of expressions that may be useful in applications involved with the variational principle. Notice the arbitrary function $S$ could be, for example, the Lagrangian.

Consider an arbitrary function,

$$S = S(u, v, t),$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$

For arbitrary increments $\Delta, \delta, \text{ and } \varepsilon$ the value of

$$S(u + \Delta, v + \delta, t + dt)$$

can be approximated by,

$$S(u + \Delta, v + \delta, t + dt) = S(u, v, t) +$$

$$+ \frac{\partial S}{\partial u_1} \Delta_1 + \frac{\partial S}{\partial u_2} \Delta_2 + \frac{\partial S}{\partial u_3} \Delta_3 +$$

$$+ \frac{\partial S}{\partial v_1} \delta_1 + \frac{\partial S}{\partial v_2} \delta_2 + \frac{\partial S}{\partial v_3} \delta_3 +$$

$$+ \frac{\partial S}{\partial t} dt$$

Sometimes the following notation is used,

$$\frac{\partial S}{\partial u} = \left( \frac{\partial S}{\partial u_1}, \frac{\partial S}{\partial u_2}, \frac{\partial S}{\partial u_3} \right)$$

Accordingly,

$$S(u + \Delta, v + \delta, t + dt) = S(u, v, t) + \Delta \cdot \frac{\partial S}{\partial u} + \delta \cdot \frac{\partial S}{\partial v} + \frac{\partial S}{\partial t} dt$$
For an arbitrary function, $S = S(u, v, t)$,

\[
\frac{dS}{dt} = \frac{\partial S}{\partial u_1} \dot{u}_1 + \frac{\partial S}{\partial u_2} \dot{u}_2 + \frac{\partial S}{\partial u_3} \dot{u}_3 + \frac{\partial S}{\partial v_1} \dot{v}_1 + \frac{\partial S}{\partial v_2} \dot{v}_2 + \frac{\partial S}{\partial v_3} \dot{v}_3 + \frac{\partial S}{\partial t}
\]

Sometimes, the following notation is used

\[
\frac{\partial S}{\partial u} \equiv \left( \frac{\partial S}{\partial u_1}, \frac{\partial S}{\partial u_2}, \frac{\partial S}{\partial u_3} \right)
\]

Accordingly,

\[
\frac{dS}{dt} = \dot{u} \cdot \frac{\partial S}{\partial u} + \dot{v} \cdot \frac{\partial S}{\partial v} + \frac{\partial S}{\partial t}
\]