Analogy between

- “electronic excitations in an atom” and the
- “mechanical motion of a forced harmonic oscillator”

How to choose the value of the corresponding spring constant $k$?

**Resonant Absorption**

Transition between energy levels

$\Delta E = k \omega_o$

**Mechanical resonance**

We identify the mechanical resonance frequency with the optical resonance frequency $\omega_o$.

$$\omega_{res} = \sqrt{\frac{k}{m_e}}$$

Let's assume we know the particular transition energy $\Delta E$. Therefore we know $\omega_o$, also called $\omega_{res}$.
How to identify the corresponding damping constant "b"?

Scattering

Charged mechanical oscillator

\[ \Delta E = \hbar \omega_0 \]

If \( \omega_f = \omega_0 \) the incident photon is absorbed by the atom

Resonant Absorption
The emission of electromagnetic waves by the accelerated charge, can be considered as a channel of "dissipative" energy.

The associated electromagnetic radiation damping $b$, in terms of the atomic physical quantities, is given in expression (25) of the QM-Lecture-3.

Light-atom interaction by a resonance absorption process, can therefore be modelled by a equation similar to a harmonically forced damped oscillator.

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F_0 \cos(\omega_f t)$$

- $m$ is the mass of the electron
- $x$ is the "position" of the electron
- $\omega_f$ is one of the atom's discretized absorption frequencies
- $b = b(\omega_f, m, ...)$
- $k= m \omega_f^2$
- $F_0 = e E_0$, external electric field amplitude
- $\omega_o, k$
- External excitation of driving frequency $\omega_f$
Solving the forced harmonic oscillator equation, Eq. 1, in complex variable

First, we apply a trick:

Let \( y = y(t) \) be the solution of

\[
    m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = F_0 \sin(\omega_f t) \tag{2}
\]

Multiply Eq. 2 by \( i \)

\[
    m \frac{d^2 (iy)}{dt^2} + b \frac{d(iy)}{dt} + k(iy) = F_0 [i \sin(\omega_f t)] \tag{3}
\]

Eqs. 1 + 3 gives

\[
    m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + kz = F_0 e^{i\omega_f t} \tag{4}
\]

where \( z(t) = x(t) + iy(t) \)

If we find a solution in Eq. 4, let's say \( z(t) \)
then a solution to Eq. 1 can be obtained by taking \( x(t) = \text{Re} \{ z(t) \} \)
Solving Eq. 4

Since the driving force is harmonic of frequency \( \omega_f \), we guess the displacement \( z(t) \) will also be harmonic of frequency \( \omega_f \). Potentially, there may be phase difference between \( F_0 e^{i \omega_f t} \) and \( z(t) \).

Thus, we propose a solution of the form,

\[
z(t) = A e^{i(\omega_f t + \phi)}
\]

where \( A \) and \( \phi \) can depend on \( \omega_f \) and other parameters.

Eq. in 4 gives,

\[
(-\omega_f^2 + \omega_0^2 + i \frac{b}{m} \omega_f)A e^{i\phi} = \frac{F_0}{m}
\]

\[
\Rightarrow A e^{i\phi} = \frac{1}{(-\omega_f^2 + \omega_0^2 + i \frac{b}{m} \omega_f)} \frac{F_0}{m}
\]

\[
= \frac{(-\omega_f^2 + \omega_0^2) - i \frac{b}{m} \omega_f}{(-\omega_f^2 + \omega_0^2)^2 + (\frac{b}{m} \omega_f)^2} \frac{F_0}{m}
\]
Notice, the complex number in the numerator can be expressed as,

\[
(-\omega_f^2 + \omega_0^2) - i \frac{b}{m} \omega_f = 
\]

\[
= \left[(-\omega_f^2 + \omega_0^2)^2 + \left(\frac{b}{m} \omega_f\right)^2\right]^{1/2} e^{i \tan^{-1} \frac{-\frac{b}{m} \omega_f}{-\omega_f^2 + \omega_0^2}} 
\]

\(\omega_f\) variable driving frequency

\(\omega_0\) oscillator natural frequency (fixed value)

\[\Rightarrow\]

\[
A e^{i \phi} = \frac{F_0/m}{\left[(-\omega_f^2 + \omega_0^2)^2 + \left(\frac{b}{m} \omega_f\right)^2\right]^{1/2}} e^{i \tan^{-1} \frac{-\frac{b}{m} \omega_f}{-\omega_f^2 + \omega_0^2}}
\]

\(A(\omega_f) = \frac{F_0/m}{\left[(-\omega_f^2 + \omega_0^2)^2 + \left(\frac{b}{m} \omega_f\right)^2\right]^{1/2}}\)

\(\phi(\omega_f) = \tan^{-1} \frac{-\frac{b}{m} \omega_f}{-\omega_f^2 + \omega_0^2}\)
Summary.
\[ m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + \kappa z = F_0 e^{i \omega_f t} \]

This equation admits solutions of the form
\[ z(t) = A e^{i(\omega_f t + \phi)} \]

where \( A = A(\omega_f) \) and \( \phi = \phi(\omega_f) \)
are given in expression (5)

Accordingly,
The solution to \( \ddot{x} \)
\[ m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + \kappa x = F_0 \cos(\omega_f t) \]
is given by
\[ x(t) = \text{Re} \{ 2(t) \} \]
\[ = A(\omega_f) \cos(\omega_f t + \phi) \]
Graphic analysis of the solutions (phasors)

Phasor force: \( F_0 e^{i \omega_f t} \)

Phasor position: \( A e^{i (\omega_f t + \phi)} \)

Phasor velocity: \( A \omega_f e^{i (\omega_f t + \phi + \pi/2)} \)

Notice:
Since \( \phi \) is always negative, the position phasor always lags the force phasor.
$\omega_f$ variable driving frequency

$\omega_o$ oscillator natural frequency (fixed value)

Variation of $\tau$

At low $\omega_f$

$$\frac{-\omega_f^2 + \omega_o^2}{2} \rightarrow -\frac{b}{m} \omega_f$$

At higher $\omega_f$ but $\omega_f < \omega_o$

$\omega_f = \omega_o$

$\phi \approx 90^\circ$

At $\omega_f > \omega_o$

$$(-\omega_f^2 + \omega_o^2) \rightarrow \phi$$
\[ A_{\text{max}} = \frac{F_0/k}{A \left(1 - \frac{1}{4A^2}\right)^{1/2}} \]

For \( Q \gg 1 \)
\[ A_{\text{max}} \propto Q \frac{F_0}{\kappa} \]

\[ \omega'_0 = \sqrt{\omega_0^2 - \frac{1}{2} \left(\frac{b}{m}\right)^2} \]

\[ = \omega_0 \sqrt{1 - \frac{1}{2Q^2}} \quad (Q = \frac{m}{b} \omega_0) \]

\[ \omega'_0 \approx \omega_0 \text{ for high } Q \]