The Maxwell Equations in Vacuum

A. Maxwell Eqs expressed in INTEGRAL form
   Line integral
   Surface integral

B. Maxwell Eqs. expressed in DIFFERENTIAL form
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A. The Maxwell Equations in Integral Form

The concept of line and surface integrals

**Line integral**

\[ \int_{\Gamma} E_t \, dx = \int_{\Gamma} \vec{E} \cdot d\vec{l} \]

**Surface integral**

\[ \int_{S} E_n \, dA = \int_{S} \vec{E} \cdot \hat{n} \, dA = \int_{S} \vec{E} \cdot d\vec{A} \]

\[ d\vec{A} = \hat{n} \, dA \]

where

\( \hat{n} \) is the unit vector perpendicular to the patch.
The First Maxwell Eq.

\[ \int_{S} \vec{E} \cdot d\vec{A} = \frac{Q_{\text{inside}}}{\varepsilon_0} \]

\( \varepsilon_0 \): permittivity of free-space

\( \vec{E} \) is evaluated at each point on the surface \( S \).

Electric flux \( \Phi_E = \int_{S} \vec{E} \cdot d\vec{A} \)

Notice:

If an electric field line originates at a given point \( P \), then there must exist charge at that point.

Proof:

Obviously \( \Phi_E \neq 0 \)

Therefore, according to the 1st ME, there must be charge at \( P \)
This is what happens in metals, for example.

When charges are present:
- Lines of $\vec{E}$ originate on positive charges and terminate on negative charges.
- Everywhere else, the $\vec{E}$ lines can twist and turn in space, but they cannot start or stop.

The Second Maxwell Eq.

$$\int_B \cdot d\vec{A} = 0$$

No magnetic monopoles have been observed (so far)
The Third Maxwell Eq.

The motion of the magnet produces an induced current along the coil.

The existence of a current implies the presence of an electric field or (equivalently) an electromotive force.

\[ E = \oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} \quad (= Ri) \]

But, what is the value of \( E \)?
(\( 1v, 2v, -2.3v \ldots ? \))

\[ E = -\frac{d}{dt}\phi_m \]

where \( S \) is any open surface having the loop \( \Gamma \) as its boundary.
Notice:
A $\mathbf{B} = \mathbf{B}(t)$ ensures the induction of an electric field $\mathbf{E}$ and, consequently, a current $i$ along a coil.

However, a static magnetic field $\mathbf{B}_0$ can also be exploited to generate currents along a coil (see figure).

$\mathbf{A}$ revolves around $\mathbf{B}_0$, causing a time dependent magnetic flux $\Phi_m$.

So, to have an electromotive force what matters is to have time dependent $\Phi_m(t)$.

In the figure above, since there is not an induced electric field, how can we have an electromotive force $\mathbf{E} = \oint \mathbf{E} \cdot d\mathbf{s}$? Shouldn't $\mathbf{E}$ be zero because $\mathbf{E} = \mathbf{0}$?

Answer: We better re-define $\mathbf{E}$ as

$$\mathbf{E} = \oint \mathbf{F} \cdot d\mathbf{l}$$

where magnetic electric force $\mathbf{F}$ on $\mathbf{q}$.
The Fourth Maxwell Eq.

\[
\mathcal{E} = \oint \frac{\text{magnetic force on } \vec{q}}{\text{loop } \Gamma} \cdot d\vec{s} \\
= \oint \frac{\vec{q} \times \vec{B}_0}{\text{} \cdot d\vec{s}} \\
= \oint (\vec{\nabla} \times \vec{B}_0) \cdot d\vec{s} \\
\]

Maxwell noticed this law was incomplete.
James Clerk Maxwell noticed there was something wrong with the fourth equation. For example:

\[ \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \oint \mathbf{J} \cdot d\mathbf{l} \quad \text{(current crossing the surface } S_1) \]

\[ B = \frac{\mu_0}{2\pi r} i \quad \text{(1)} \]

On the other hand,

\[ \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \oint \mathbf{J} \cdot d\mathbf{l} \quad \text{(current crossing the surface } S_2) \]

\[ B = \mu_0 \times 0 \quad \text{(2)} \]

How can it be possible that (1) and (2) give different results for the same \( B \)? Something must be wrong with the Ampère's law.
In order to solve this contradictory situation, let's take a look at what is going on inside the parallel plates capacitor.

Let's remember
\[ E = \frac{Q}{A \varepsilon_0} \]

(A is the area of the plates)

The current \(i\) continuously accumulates charge on the plates, this is \(Q = Q(t)\)

Notice, the electric flux \(\Phi_e\) crossing the surface \(S_2\) is given by
\[ \Phi_e = EA = \frac{Q}{\varepsilon_0} \]
or
\[ Q = \varepsilon_0 \Phi_e \]

From the last expression, we obtain
\[ \frac{dQ}{dt} = \varepsilon_0 \frac{d}{dt} \Phi_e \]

but, this is equal to \(i\)

\[ i = \varepsilon_0 \frac{d}{dt} \Phi_e \]

So, an electric flux that changes with time is "equivalent" to a current.
Maxwell proposed the following modified version of the Ampere's law:

\[ \int_{\text{loop } \Gamma} \mathbf{B} \cdot d\mathbf{s} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \]

on, written more explicitly:

\[ \int_{\text{loop } \Gamma} \mathbf{B} \cdot d\mathbf{s} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d}{dt} \int_{\text{surface } S} \mathbf{E} \cdot d\mathbf{A} \]

called "displacement current"

\[ i_d \]

Verification: Let's go back to our previous example and apply the "new" 4th ME and find out \( B \) using surface \( S_1 \)

No electric field \( E \) crossing the surface \( S_1 \) so the 4th ME takes the form:

\[ \int_{\text{loop } \Gamma} \mathbf{B} \cdot d\mathbf{r} = \mu_0 i \]

From which we obtain:

\[ B = \frac{\mu_0 i}{2 \pi r} \]
What about if we choose the surface $S_2$?

No current crossing the surface $S_2$, but, in this case, there does exist an electric field $\vec{E}$ crossing the surface. So, the 4th Maxwell equation takes the form:

$$\int \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{A}$$

$$= E A = \frac{Q}{\lambda \varepsilon_0} A = \frac{Q}{\varepsilon_0}$$

$$= \mu_0 \varepsilon_0 \frac{d}{dt} \left( \frac{Q}{\varepsilon_0} \right) = \mu_0 \frac{dQ}{dt} = \mu_0 i$$

$$\int \vec{B} \cdot d\vec{l} = \mu_0 i$$

From which we obtain:

$$B = \frac{\mu_0}{2\pi r} i$$  \hspace{1cm} \text{same result as when the surface } S_1 \text{ was used. (4)}$$
Fourth Maxwell Equation

\[ i = \frac{dQ}{dt} \text{ charge on the plate} \]

\[ i_d = \varepsilon_0 \frac{d\Phi_E}{dt} \text{ "displacement current"} \]

More general:

\[ \int_{\Gamma} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_{S} \mathbf{j} \cdot d\mathbf{A} + \mu_0 \varepsilon_0 \frac{d}{dt} \int_{S} \mathbf{E} \cdot d\mathbf{A} \]

Fourth Maxwell Equation
B. The Maxwell Equations in Differential Form

First, some definitions in vector algebra:

- The operator $\nabla$ (called "gradient")

$$\nabla \equiv \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

- $\nabla$ acts on scalar fields

Example

If $\phi$ is the electric potential,

$$\nabla \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$$

which, as we know, gives

$$=-E_x, -E_y, -E_z$$

Thus,

$$\nabla \phi = -\vec{E}$$

We see, when $\nabla$ acts on a scalar field it gives a vector
The divergence operator "\( \nabla \cdot \)"

Given a vector field \( \vec{E} \),

\[
\nabla \cdot \vec{E} \equiv \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}
\]

\( \nabla \cdot \vec{E} \) is a scalar quantity.

The rotational operator "\( \nabla \times \)"

Given a vector field \( \vec{E} \),

\[
\nabla \times \vec{E} \equiv \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
E_x & E_y & E_z
\end{vmatrix}
\]

\[
\nabla \times \vec{E} = \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \vec{i} - \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \vec{j} + \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \vec{k}
\]

\( \nabla \times \vec{E} \) is a vectorial quantity.
Flux of a vector field across a surface $S$

Below we will generalize this result to the case in which the macroscopic volume is divided into multiple infinitesimal volumes.

**Reference:** R. Feynman, "The Feynman Lectures on Physics," Vol-II, Chapter 2 (Differential Calculus of Vector Fields;) Chapte 3 (Vector Integral Calculus.)
The Gauss' theorem

First, let's calculate the flux across a surface containing an infinitesimal volume $\Delta V$

$$\vec{E} = (E_1, E_2, E_3)$$

What is the flux through the cube of volume $\Delta x \Delta y \Delta z$?

$$\Phi_{\text{cube}} = \Phi_1 + \Phi_2 + \ldots + \Phi_6$$

$$\Phi_1 = -E_1(x) \Delta y \Delta z$$

$$\Phi_2 = E_1(x + \Delta x) \Delta y \Delta z$$

$$\Phi_1 + \Phi_2 = [E_1(x + \Delta x) - E_1(x)] \Delta y \Delta z$$

$$= [\frac{\partial E_1}{\partial x} \Delta x] \Delta y \Delta z$$

Similar expressions for $\Phi_3 + \Phi_4$ and $\Phi_5 + \Phi_6$

Thus,

$$\Phi_{\text{cube}} = [\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} + \frac{\partial E_3}{\partial z}] \Delta x \Delta y \Delta z$$

$$= [\nabla \cdot \vec{E}] \Delta V$$

Flux through the cube of volume $\Delta V = \Delta x \Delta y \Delta z$

(6)
On the other hand, let's generalize the result given in (5)

\[ \Phi_s = \int_s \vec{E} \cdot d\vec{A} \]  \hspace{1cm} \text{surface integral} \hspace{1cm} (7)

Notice, the volume inside the surface \( S \) can be divided into many very small cubes. (We just need to make them very tiny; infinitesimal)

Each cube \( i \) has a surface \( A_i \) and a volume \( \Delta V_i \)

Using (5)

\[ \Phi_s = \Phi_{s_1} + \Phi_{s_2} + \ldots \] \hspace{1cm} (8)

Using (6)

\[ = (\nabla \cdot \vec{E}) \Delta V_1 + (\nabla \cdot \vec{E}) \Delta V_2 + \ldots \]

\( \) which is nothing but

\[ = \int_V \nabla \cdot \vec{E} \, dV \leftarrow \text{volume integral} \hspace{1cm} (9) \]

From (7) and (9)
The Gauss theorem

\[ \int_S \mathbf{E} \cdot d\mathbf{A} = \int_V \nabla \cdot \mathbf{E} \, dv \]

**S** is any mathematical closed surface

**V** is the volume inside **S**

Reference: R. Feynman, "The Feynman Lectures on Physics," Vol-II, Chapter 3 (Section 3-3 The Gauss theorem.)

Example. Expressing the charge conservation principle in differential form

\[ \mathbf{j} = \frac{\text{current}}{\text{area}} \]

\( \mathbf{j} \) points in the direction of the local velocity \( \mathbf{v} \) of the charges

Current crossing \( d\mathbf{A} \) = \( \mathbf{j} \cdot d\mathbf{A} \)

in units of coulomb/sec
\[ I = \text{net current crossing the surface } S \]
\[ = \int_S \vec{j} \cdot d\vec{A} \]

I must be equal to the change per unit time of the net charge inside the volume of the surface \( S \)
\[ I = -\frac{d}{dt} Q_{\text{inside}} \]
\[ = -\frac{d}{dt} \int_V \rho \, dv \]

Thus,
\[ \int_S \vec{j} \cdot d\vec{A} = -\frac{d}{dt} \int_V \rho \, dv \] (using Gauss' theorem)

where \( \rho \) is the charge density (coulomb/m^3)
\[ \rho = \rho(x, y, z, t) \]

\[ \int_S \nabla \cdot \vec{j} \, dV = -\frac{d}{dt} \int_V \rho \, dv \rightarrow -\int_V \frac{2 \rho}{\delta t} \, dv \] since the surface \( S \) is stationary
Since the surface \( S \) is arbitrary (it could even be infinitesimal), it must be that:

\[
\nabla \cdot \vec{J} + \frac{2}{\varepsilon_0} \frac{\partial \rho}{\partial t} = 0
\]

Conservation of charge expressed in differential form

Example. Expressing the FIRST Maxwell Eq. in differential form

\[
\int_S \vec{E} \cdot d\vec{a} = \frac{Q_{\text{inside}}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_V \rho \, dv
\]

Applying Gauss' theorem

\[
\int_V \nabla \cdot \vec{E} \, dv = \int_V \frac{\rho}{\varepsilon_0} \, dv
\]

Since this is valid for any volume, including an infinitesimal one, it must be that:

\[
\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}
\]

First's Maxwell Eq. in differential form

Example. Expressing the SECOND Maxwell Eq. in differential form

\[
\int_S \vec{B} \cdot d\vec{a} = 0
\]
Applying Gauss theorem

\[ \int_{V} \nabla \cdot \mathbf{B} \, dv = 0 \quad \text{for any arbitrary volume } V \]

This implies

\[ \nabla \cdot \mathbf{B} = 0 \]

Second Maxwell's Eq in differential form
The Stoke's Theorem

About the circulation of a vector field $\vec{E}$

Circulation of $\vec{E}$ is the line integral of the tangential component of $\vec{E}$ around the loop $\Gamma$.

$$\oint_{\Gamma} \vec{E} \cdot d\vec{l} = \iint_{\text{area}} \nabla \times \vec{E} \, dA$$

Now, the loop $\Gamma$ has been divided into two contiguous loops $\Gamma_1$ and $\Gamma_2$.

Notice:

$$\oint_{\Gamma} = \oint_{\Gamma_1} + \oint_{\Gamma_2}$$
Generalization

\[ \mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2 + \ldots \]

Here we have assumed that the different loops \( \Gamma_1, \Gamma_2, \ldots \) are in the plane of \( \Gamma \).

But it doesn't have to be that way.

The small loops can lie on any surface having \( \Gamma \) as its boundary.

By choosing the loops small enough, each can be considered a flat rectangular loop.

Circulation around a rectangular loop

For the particular small rectangular loop shown in the figure, let's choose a reference such that the loop results lying at the XY plane.
If the result of the circulation can later be put in vectorial notation, then it will be the same no matter how the axis XYZ were chosen:

\[ \int \vec{E} \cdot d\vec{s} = E_x(1) \Delta x + E_y(2) \Delta y + \]

\[ -E_x(3) \Delta x - E_y(4) \Delta y \]

\[ \text{Since } E_x(3) - E_x(1) \approx \frac{\partial E_x}{\partial y} \Delta y \]

\[ E_y(2) - E_y(4) \approx \frac{\partial E_y}{\partial x} \Delta x \]

\[ \int \vec{E} \cdot d\vec{s} = \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \Delta x \Delta y \]

The z-component of \( \nabla \times \vec{E} \).
\[ \int_{\partial} \vec{E} \cdot d\vec{s} = (\nabla \times \vec{E}) \cdot \Delta x \Delta y \]

We can express this result in vectorial form by realizing that the component in the direction perpendicular to the small loop is:

\[ = (\nabla \times \vec{E}) \cdot \hat{n} \Delta x \Delta y = (\nabla \times \vec{E}) \cdot d\vec{A} \Delta x \Delta y \]

We extend this result to multi-connected loops:

Stoke's theorem

\[ \int_{\Gamma} \vec{E} \cdot d\vec{l} = \int_{S} (\nabla \times \vec{E}) \cdot d\vec{A} \]

\( S \) is any surface whose boundary is \( \Gamma \)

Reference: R. Feynman, "The Feynman Lectures on Physics," Vol-II, Chapter 3 (Section 3-6 The Stokes theorem.)
Example. Expressing the THIRD Maxwell Eq. in differential form

\[ \int \vec{E} \cdot d\vec{A} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A} \]

Applying Stokes' theorem

If the surface \( S \) is stationary

\[ \iint (\nabla \times \vec{E}) \cdot d\vec{A} = -\iint \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} \]

This expression is valid for any arbitrary surface, including a infinitesimal rectangle. Thus

Third Maxwell Equation expressed in differential form

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

Example. Expressing the FOURTH Maxwell Eq. in differential form

\[ \int \vec{B} \cdot d\vec{A} = \mu_0 \int \vec{j} \cdot d\vec{A} + \varepsilon_0 \mu_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{A} \]

Applying Stokes' theorem

If \( S \) is stationary

\[ \iint (\nabla \times \vec{B}) \cdot d\vec{A} = \mu_0 \int \vec{j} \cdot d\vec{A} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A} \]
Since this expression is valid for any arbitrary surface $S$, then

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

Fourth Maxwell Equation expressed in differential form