3.1 INTRODUCTION

The phenomena of refraction and refraction of light at a plane surface separating two media of different dielectric properties can be classified into two aspects:

the **kinematics** properties, and

the **dynamics** properties.

![Fig. 1]

**Kinematics properties**

These include the relationship that exists between the angles of incidence $\theta_{\text{inc}}$, reflection $\theta_{\text{reflection}}$, and refraction $\theta_{\text{refraction}}$

\[ i) \quad \theta_{\text{inc}} = \theta_{\text{reflection}} \]

\[ ii) \quad n_i \sin(\theta_{\text{inc}}) = n_r \sin(\theta_{\text{refraction}}) \]

These two relations result from the fact that,

“there are boundary conditions to be satisfied; but they do not depend on the detailed nature of those boundary conditions.”

Ref. J. Jackson, Classical Electrodynamics.

For example, let’s assume that a particular boundary condition has the form

\[ A \vec{E}_{\text{inc}}(\vec{x}, t) + B \vec{E}_{\text{reflec}}(\vec{x}, t) + C \vec{E}_{\text{transm}}(\vec{x}, t) = \vec{0} \quad \text{at } z=0 \quad (1) \]
which has to be satisfied at any position $\mathbf{x}$ on the interface (see Fig. 1 above above,) and at any time $t$.

The particular values of $A$, $B$ and $C$ will depend on the particular polarization of the fields we are dealing with ($s$-polarization or $p$-polarization, which will be defined below.) $A$, $B$, and $C$ will then be determined by the Maxwell Equations.

But, whatever the relationship among $A$, $B$ and $C$ is, that does not affect the relations shown in $i)$ and $ii)$ above; those results are obtained (as we will see in the next section) simply from the fact that there are boundary conditions that need to be satisfied.

**Dynamic properties**

These include,

$iii)$ The intensities of the reflected and transmitted radiation

$iv)$ Phase changes and polarization

These properties depend on the particular nature of the boundary conditions. That is, they depend on the particular values of $A$, $B$, and $C$ that appear in expression (1) above.

**PROPAGATING PLANE WAVES IN A DIELECTRIC MATERIAL**

- To formally derive the kinematics and dynamics properties mentioned above, we will consider the propagation of plane waves.

In each dielectric medium, light propagation is governed by the wave equation,

$$\nabla^2 \begin{pmatrix} \hat{E} \\ \hat{B} \end{pmatrix} - \frac{(\varepsilon / \varepsilon_0)}{c^2} \frac{\partial^2}{\partial t^2} \begin{pmatrix} \hat{E} \\ \hat{B} \end{pmatrix} = 0$$

Or, since $c \equiv \sqrt{\frac{1}{\varepsilon_0 \mu_0}}$

$$\nabla^2 \begin{pmatrix} \hat{E} \\ \hat{B} \end{pmatrix} - \frac{1}{\left(\frac{1}{\sqrt{\mu_0 \varepsilon}}\right)^2} \frac{\partial^2}{\partial t^2} \begin{pmatrix} \hat{E} \\ \hat{B} \end{pmatrix} = 0$$

where one identifies a complex velocity,

$$v \equiv \sqrt{\frac{1}{\mu_0 \varepsilon}}$$

Or equivalently,
\[ \nabla^2 \begin{bmatrix} \vec{E} \\ \vec{B} \end{bmatrix} - \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} \begin{bmatrix} \vec{E} \\ \vec{B} \end{bmatrix} = 0 \]

(2)

where the dielectric constant \( \varepsilon / \varepsilon_0 \) will be assumed to be, in general, a complex number, and one identifies a complex velocity,

\[ v \equiv \frac{c}{n} \]

(We derived the above expressions in Section 2.2B; expression (17) of that section.)

- Equation (2) admits plane waves as solutions,

\[
\begin{align*}
\vec{E}(\vec{r}, t) &= \vec{E}_o e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\
\vec{B}(\vec{r}, t) &= \vec{B}_o e^{i(\vec{k} \cdot \vec{r} - \omega t)}
\end{align*}
\]

(3)

provided that,

\[ k^2 = \frac{\omega^2}{c^2} n^2. \]

Since \( \omega c = 2\pi f = 2\pi \lambda / \lambda_0 \) (where \( \lambda_0 \) is the wavelength of light in vacuum), \( k^2 = \left( \frac{2\pi}{\lambda_0} n \right)^2 \).

But to simplify the notation, in this chapter we will use simply \( \lambda \) for the wavelength of light in vacuum, instead of \( \lambda_0 \). That is,

\[ k^2 = \left( \frac{2\pi}{\lambda} n \right)^2 \]

(4)

We emphasize that the solutions given in (3) are valid even when \( n^2 \) (hence \( k^2 \)) is complex.

- On the other hand, the Maxwell equation \( \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{B} = 0 \), when applied to the plane waves given in (3), implies \( i \vec{k} \times \vec{E}_o - i \omega \vec{B}_o = 0 \). That is,

\[ \vec{k} \times \vec{E}_o = \omega \vec{B}_o \]

Equivalently,

\[ \frac{\vec{k} \times \vec{E}_o}{k} = \frac{\omega}{k} \vec{B}_o \]
\(\omega/k\) is the phase velocity \(v_{\text{phase}} = 1/\sqrt{\mu \varepsilon}\), which gives,

\[
\frac{\vec{k} \times \vec{E}_o}{k} = \frac{1}{\sqrt{\mu \varepsilon}} \vec{B}_o;
\]

\[
\vec{B}_o = \sqrt{\mu \varepsilon} \frac{\vec{k} \times \vec{E}_o}{k};
\]

\[
\vec{B}_o = \frac{1}{v_{\text{phase}}} \frac{\vec{k} \times \vec{E}_o}{k}
\]

(5)

For transverse waves

\[
B_o = \frac{1}{v_{\text{phase}}} E_o
\]

- Electromagnetic waves propagate energy

\[
\vec{S} = \vec{E} \times \frac{1}{\mu} \vec{B} \quad \text{energy flow per unit area per second}
\]

where \(\mu\) is the magnetic permeability of the medium.

For harmonic fields, like the one in (3), it turns out,

\[
I = \langle \vec{S} \rangle = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}} |E_0|^2 \frac{k}{k} \quad \text{time-average energy per unit area}
\]

(6)

\(I\) is called the light intensity.

Equivalent forms of (5) are,

\[
I = \langle \vec{S} \rangle = \frac{\varepsilon}{2} \frac{1}{k} |E_0|^2 \frac{k}{k} = \frac{\varepsilon c}{2n} |E_0|^2 \frac{k}{k}
\]

\[
I = \langle \vec{S} \rangle = \frac{1}{2} \frac{\mu \varepsilon}{2} \frac{1}{k} \frac{k}{k} = \frac{n}{2} \frac{\mu \varepsilon}{2} |E_0|^2 \frac{k}{k}
\]

where \(n\) is the index of refraction of the medium.
Terminology that is used to describe reflection and refraction at a dielectric-dielectric interface

We adapt the solutions given in (3) to setup the problem of reflections and refraction at an interface.

Three general plane waves are considered,

\[ \vec{E}_i = \vec{E}_{oi} e^{j(\vec{k}_i \cdot \vec{r} - \omega_t t)} \]  \hspace{1cm} \textit{Incident wave}

\[ \vec{E}_r = \vec{E}_{or} e^{j(\vec{k}_r \cdot \vec{r} - \omega_r t)} \]  \hspace{1cm} \textit{Reflected wave}

\[ \vec{E}_t = \vec{E}_{ot} e^{j(\vec{k}_t \cdot \vec{r} - \omega_t t)} \]  \hspace{1cm} \textit{Transmitted wave}

Here \( \vec{r} = (x, y, z) \)

Or equivalently, if we want to manipulate just real variables

\[ \vec{E}_i = \vec{E}_{oi} \cos(\vec{k}_i \cdot \vec{r} - \omega_t t) \]

\[ \vec{E}_r = \vec{E}_{or} \cos(\vec{k}_r \cdot \vec{r} - \omega_r t) \]

\[ \vec{E}_t = \vec{E}_{ot} \cos(\vec{k}_t \cdot \vec{r} - \omega_t t) \]  \hspace{1cm} \textit{(8)}

We need to find out the relationship among the \( \vec{k} \) vectors (so far we are assuming they have arbitrary orientations, \( n_i \)) as well as the relationship among the amplitudes \( \vec{E}_{oi}, \vec{E}_{or} \), and \( \vec{E}_{ot} \).

We write the corresponding \( \vec{k} \) vectors more explicitly,
\[ \vec{k}_i = (k_{ix}, k_{iy}, k_{iz}) \]
\[ \vec{k}_r = (k_{rx}, k_{ry}, k_{rz}) \]
\[ \vec{k}_t = (k_{tx}, k_{ty}, k_{tz}) \]  

(8)

According to (3), and considering that \( \vec{k}_i \) and \( \vec{k}_r \) are in the same medium, we have,

\[ |\vec{k}_i| = |\vec{k}_r| = \frac{2\pi}{\lambda} = \frac{2\pi}{\lambda} n_i \]

\[ |\vec{k}_t| = \frac{2\pi}{\lambda} \frac{1}{n_t} \]  

(9)

\( n_i \) and \( n_t \) are the indices of refraction of the two media, respectively. \( \lambda \) is the wavelength of the incident light when in vacuum.