GRATINGS and SPECTRAL RESOLUTION

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[NOTE: The terminology about phases is repeated from the precedent lecture notes. You may want to jump to Section 1.5]

Let’s consider a fully illuminated grating of length \( L \) and composed of \( N \) lines equally separated by a distance “\( d \)”

\[
k = \frac{2\pi}{\lambda}
\]

\( N \) lines separated by a distance “\( d \)” from each other

**Fig. 1** For radiation of a given wavelength, several maxima of intensity (\( m=0, 1, 2,... \)) are observed at different angular positions as a result of constructive interference by the \( N \) lines sources from the grating.
1. Calculation of the maxima of interference by the method of phasors

1.1 Case: Phasor addition of two waves

Wave-1 = A \cos(kx) \quad \rightarrow \quad \text{phasor-1} = \Psi_1 = Ae^{i[kx]}

Wave-2 = A \cos(kx+k\delta) \quad \rightarrow \quad \text{phasor-2} = \Psi_2 = Ae^{i[kx+k\delta]}

\[ A \cos(kx) = \text{Real}(\Psi_1) \]

\[ A \cos(kx+k\delta) = \text{Real}(\Psi_2) \]

Notice,

A path difference of \delta gives a phase difference of \[ k \delta = \frac{2\pi}{\lambda} \delta \]

\[ \lambda \quad \rightarrow \quad k \delta = 2\pi \]

*Fig. 2* Real-variable waves are represented by their corresponding complex-variable phasors.

Phasors:

\[ \Psi = \Psi_1 + \Psi_2 = Ae^{i[kx]} + Ae^{i[kx+k\delta]} \]

Real variable:

\[ A \cos(kx) + A \cos(kx+k\delta) = \text{Real}(\Psi_1 + \Psi_2) = \text{Real}(\Psi) \]
1.2 Case: Phasor addition of \( N \) waves

\[
\Psi = \Psi_1 + \Psi_2 + \ldots + \Psi_n = \\
= Ae^{ikx} + Ae^{i(kx+k\delta)} + Ae^{i(kx+2k\delta)} + \ldots + Ae^{i(kx+(n-1)k\delta)} \\
= e^{ikx}[A + Ae^{ik\delta} + Ae^{i2k\delta} + \ldots + Ae^{i(N-1)k\delta}] \\
\Psi = e^{ikx}\left[ \sum_{s=0}^{N-1} Ae^{isk\delta} \right]
\]

(13)

Fig. 4 shows the graphic interpretation of the result (13).

1.3 Condition for constructive interference

This occurs when contiguous interfering waves travel a path difference \( \delta \) exactly equal to an integer number of wavelengths

\[
\delta = m \lambda \quad (m = 0,1,2,3,\ldots);
\]

(14)
which take place at the angles $\theta = \theta_m$ that satisfies:

$$\delta \equiv d \sin \theta_m = m \lambda$$

Equivalently, since $k = 2\pi/\lambda$, the max of constructive interference occurs when the phase difference $k \delta$ between two contiguous waves is exactly equal to an integer number of $2\pi$;

$$k \delta = m 2\pi$$

Condition for max of constructive interference

where $(m = 0, 1, 2, 3, \ldots)$. 

Fig. 5 Condition in which the phase difference between two contiguous phasors is $m 2\pi$. They contribute to the max. of order m (see also Fig. 6.12 below).
Fig. 6 Condition to produce a maximum of intensity (of order m). **Top:** All the N phasors add up as to give a phasor of the largest possible amplitude. **Bottom:** Visualization of the corresponding maxima of order m (m=0, 1, 2, ...).

1.4 Evaluating the sharpness of the peaks

1.4A Condition for evaluating the sharpness of the “zero-th order” peak.

Around a peak of maximum intensity, the intensity drops to zero, thus defining a line thickness. We would like to calculate the angular broadening associated to that thickness of line intensity.

Let's consider first the max of order \( m=0 \). Using the method of phasors, we realize that,

A path difference (between contiguous radiators) gives a phase difference of

\[
\delta \quad \rightarrow \quad k \delta = \frac{2\pi}{\lambda} \delta
\]

If \( \delta = \frac{\lambda}{N} \),

\[
k \left( \frac{\lambda}{N} \right) = \frac{2\pi}{N}
\]

For \( N \) oscillators, the total accumulated phase would be \( 2\pi \);

But Fig. 4 indicates that when the total accumulated phase is \( 2\pi \) the phasor sum would be zero. In other word, an intensity of zero amplitude wave would be obtained if

\[
\delta = \frac{\lambda}{N} \quad \text{Condition for zero intensity} \quad (15)
\]

Since \( \delta \equiv d \sin(\theta) \), expression (15) determines the angle \( \theta_{\text{min}} \) that locates the minimum of intensity (next to the maximum of order \( m=0 \)),

\[
d \sin (\theta) = \frac{\lambda}{N}
\]
or

$$N \cdot d \sin (\theta_{\min,0}) = \lambda$$  \hspace{1cm} \text{(16)}$$

Condition for the angular position (next to the maximum of order \(m=0\)) for which the light intensity is zero.

\[\text{Fig. 7} \text{ Condition for finding the angular position of the minimum of intensity closest to the maximum of order } m=0 \text{ (hence, giving a measurement of the angular sharpness of the peak of order } m=0)\]

1.4B Condition for evaluating the sharpness of the \("m\>-th order\) peak.

The angular location \(\theta_{\min, m}\) of the minimum of intensity closest to the \(m\)-th order maximum of intensity can be calculated in a similar fashion. Indeed, in the case \(m=0\) we chose a path difference of \(\delta = \lambda / N\) between two contiguous waves.

For the case \(m \neq 0\) let’s choose

$$\delta = m \lambda + \lambda / N$$
(i.e. a path difference a bit bigger than the one that gives the peak of order $m$).

The corresponding phase rotation is

$$k\delta = \frac{(2\pi/\lambda)}{m\lambda + \lambda/N} = m2\pi + 2\pi/N.$$  

Notice, however, that the latter rotation is equivalent to a net rotation of $2\pi / N$ (since a rotation of $m2\pi$ brings the phasor to the same place.)

$$k\delta \iff 2\pi/N.$$  

Thus we are choosing a net phase difference between two contiguous waves equal to $2\pi / N$.

Since we have $N$ phasors, after $N$ rotations we will have completed a full rotation, which brings the total phasor to zero. This is illustrated in Fig. 14.

\[\text{\textbf{Fig. 8}}\]  
For a given line-space $d$ and $\lambda$, the diagrams (top and bottom figures) show the angle $\theta_{\text{max},m}$ that determines the intensity-maximum of $m$-th order ($d \sin \theta_{\text{max},m} = m \lambda$). As the angle increases a bit, a minimum of intensity will occur at the angle $\theta = \theta_{\text{min},m}$, which is determined by the condition $d \sin(\theta_{\text{min},m}) = m \lambda + \lambda/N$ (see also Fig 6.14 below.).
**Fig. 9** The diagram shows the condition that determine the angle \( \theta = \theta_{\text{min}; m} \) (closest to the angle \( \theta = \theta_{\text{max}; m} \)) for which the intensity at the far located screen is zero. This occurs when \( d \sin \theta = m \lambda + \lambda / N \)

The condition
\[
\delta \equiv d \sin(\theta_{\text{max}, m}) = m \lambda
\]  
(17)
gives the angular location of the max of order \( m \).

The condition
\[
\delta' = m \lambda + \lambda / N
\]
(18)

or equivalently,
\[
N \delta' = N d \sin(\theta_{\text{min}, m}) = m N \lambda + \lambda
\]
gives the angular location of the minimum immediately next to the max of order \( m \).

From (17) and (18),
\[
Nd \sin(\theta_{\text{min}, m}) - N d \sin(\theta_{\text{max}, m}) = [m N \lambda + \lambda] - [N m \lambda]
\]
\[
Nd [\sin(\theta_{\text{min}, m}) - \sin(\theta_{\text{max}, m})] = \lambda
\]
The left side is approximately equal to,
\[
Nd \left[ \cos \theta_{\text{max}, m} \right] \Delta \theta = \lambda
\]
where \( \Delta \theta = (\theta_{\text{max}, m} - \theta_{\text{min}, m}) \equiv \Delta \theta_{\text{width}; m} \)

\[
\Delta \theta_{\text{width}; m} = \lambda / [Nd \cos \theta_{\text{max}, m}] \quad \frac{1}{2} \text{ width of the peak of order } m
\]  
(19)
1.5 Gratings and Spectral Resolution

The figure shows incident radiation of two different wavelengths. We want to evaluate the ability of a given grating to distinguish them.

![Diagram of gratings](image)

*Fig. 11 Radiation of different wavelength (or frequency) produces maxima of intensity at different angular positions*

Each radiation will produce its own independent spectrum (a set of peaks of different orders). Let’s focus on a particular order, let’s say the order $m$.

We know that the angular location of the peak increases with $\lambda$, since $d\sin(\theta_{\text{max},m}) = m\lambda$.

Accordingly, if the value of $\lambda_2$ is too close the value of $\lambda_1$ then their $m$-order peaks of intensity (which have some width) may superimpose to the point that it would be hard to distinguish them from each other in the recorded spectrum.

How close are $\lambda_2$ and $\lambda_1$ are allowed to be and still be able to distinguishing from each other in the grating spectrum?

**Raleigh’s criterion of resolution** offers a standard metric:

The smallest difference $\Delta\lambda = \lambda_2 - \lambda_1$ allowed for this two wavelength is the one for which the angular position of minimum of intensity for $\lambda_1$ (the minimum closest to its maximum of order $m$) occurs at the same angular position of the $m$-order $\lambda_2$.

The Raleigh’s criterion is illustrated in Fig. 6.16 for the case $m=1$. 

Fig. 12 Raleigh’s criterion of resolution illustrated for the case $m=1$. We want $\theta_{\text{max},m} = \theta_{\text{min},m}$.

For the same angle $\theta$, we want to have

i) a maximum of order $m$ for the incident light of wavelength $\lambda_2$

$$d \sin(\theta) = m \lambda_2 \quad \text{or} \quad N d \sin(\theta) = N m \lambda_2$$  \hspace{1cm} (21)

ii) a minimum of intensity for the wavelength $\lambda_1$ (a minimum right after its max of order $m$)

$$N d \sin(\theta) = m N \lambda_1 + \lambda_1$$  \hspace{1cm} (22)

(21) and (22) imply,

$$N m \lambda_2 = N m \lambda_1 + \lambda_1.$$

Using $\Delta \lambda = \lambda_2 - \lambda_1$ the Raleigh’s criterion implies,
Light containing radiation of wavelength $\lambda$ is incident on a grating of $N$ lines, producing a spectral line of order $m$.

If the incident light contains radiation of wavelength $\lambda'$ that differs from $\lambda$ by a quantity $\Delta\lambda$, it will be distinguished (according to the Raleigh criteria) if $\Delta\lambda$ satisfies,

$$\Delta\lambda \geq \frac{\lambda}{Nm}$$

(23)

Equivalently, to distinguish the presence of two waves whose wavelength difference is $\Delta\lambda$, requires a grating of the following number of lines

$$N \geq \frac{1}{m} \frac{\lambda}{\Delta\lambda} \quad \text{(when using the maxima of $m$-th order)}$$

(24)

Since $\frac{\Delta\lambda}{\lambda} = -\frac{\Delta v}{v}$, this expression can be written also as

$$\Delta v \geq \frac{v}{mN} \equiv (\Delta v)_{mn}$$

(25)

This expression gives the minimum frequency-difference between two waves that a grating of $N$ lines is able to distinguish as, in fact, two distinct incident frequencies (when working around the maximum of order $m$).

1.6 Condition for the minimum length time $\Delta t$ required for the measurement of the energy $E$ with a resolution $\Delta E$.

- Consider a grating of $N$ lines (separated by a distance $d$ from each other).
  Let $\lambda = \lambda_1$ (or equivalently a given frequency $\nu = \nu_1$) for which the grating produces a maximum of intensity of order $m$ (which occurs at an angular orientation $\theta_{\text{max},m}$ given by $d \sin(\theta_{\text{max},m}) = m\lambda_1$).

Notice in the graph below that in order to fully exploit the coherent interference from the $N$ oscillators we have to require that the spatial extent of the incident wave-train exceed a given minimum length $l = AQ = mN\lambda$. 

Fig. 13 Given a grating composed of N lines, for the N oscillators to participate in the m-th order coherent interference the spatial extent of the incident wave-train radiation has to exceed a distance $\ell = mN\lambda_1$. (Otherwise, a shorter wave-train will exploit only a fraction of the N lines in the grating).

- Now, we notice something mathematically curious. The time $T_{AQ}$ needed by the light to travel the distance AQ is,

$$T_{AQ} = \frac{AQ}{c} = \frac{Nm\lambda}{c} = \frac{Nm}{\nu}$$

Using the result in (25), the expression above for $T_{AQ}$ can also be written in terms of $\Delta\nu_{\text{min}}$ (the smallest frequency difference the grating can resolve),

$$T_{AQ} = \frac{1}{\nu/(Nm)} = \frac{1}{\Delta\nu_{\text{min}}}$$  \hspace{1cm} (24)

- This mathematically curious result, lead us to an important experimental interpretation, as illustrated in Fig. 6.19. In order to resolve the frequency content of the incident radiation with a resolution $\Delta\nu$, the measurement has to last at least a time $T_{AQ} = 1/\Delta\nu$, or greater. That is,

$$(\Delta t)_{\text{measurement}} \geq \frac{1}{\Delta\nu}$$  \hspace{1cm} (25)
(Otherwise, if the incident radiation lasted shorter than \( l = mN\lambda_1 \), then not all the N lines would participate and the effective resolution of the grating would be less).

**Fig. 6.14** Aiming to distinguish the two frequencies \( \nu_1 \) and \( \nu_2 \) (\( \Delta \nu = \nu_2 - \nu_1 \)) a setup to detect the maximum of order \( m \), is chosen. Such constructive interference requires the simultaneous participation of the N wave components in the interfering wavefront CQ. As illustrated also by Fig. 6.18 above, the formation of such wavefront CQ requires a pulse duration \( \Delta t \) of greater than, or at least equal to, the time required by the light to travel the distance AQ, which is \( \Delta t = \frac{AQ}{c} \). It turns out \( \frac{AQ}{c} = \frac{1}{\Delta \nu} \).

We have arrived to a very interesting interpretation:

*In order to find out whether an incident radiation contains harmonic wave components whose frequencies differ by \( \Delta \nu \), the minimum time duration of the pulse has to be \( \frac{1}{\Delta \nu} \).*

(26)

The less uncertainty we want to have concerning the frequency components in the pulse, the longer pulse we need (i.e. a longer measurement-time will be required.)

We would need more (measurement) time if we want to find out the spectral content with more precision (i.e. with less uncertainty).
There is a close relationship between the precision $\Delta \nu$ with which we want to know the spectral content of a pulse, and the minimum time duration $T$ that the pulse is required to have in order a measurement with such spectral precision: $T_{\text{min}} = 1/\Delta \nu$.

This is also illustrated in the Fig. 6.19 where it is assumed to know the frequency $\nu_1$ is contained in the pulse $f$ (because it produces a big max of interference.)

![Short pulse](image)

It is not possible to know whether or not frequency-components from this frequency range are contained in the pulse $f$.

Notice, the results just described have nothing to do with quantum mechanics. It is a property of any wave.

It occurs, however, that quantum mechanics associate a wave character to the particles; hence the variables associated to the wave-particle (momentum, position, energy) become subjected to the frequency/time of position/momentum uncertainties.

For example, applying the concept of quantum mechanics $E = h\nu$, expression (25) can be expressed as,

$$ (\Delta E)(\Delta t)_{\text{measurement}} \geq h $$  \hspace{1cm} (27)
More examples. Uncertainty principle and the resolving power of a microscope.

Image formation and the resolving power of a lens

- P focuses at T
  A condition for a clear focused image formation is that rays take an equal time to travel from the source point to the image point.

- A point P' will focus at a region very close to T.
  If P' is too close to P, their image will superimpose and become too blur to distinguish them; they would appear to be the same point.
  How close is too close?

\[ P' \]

\[ \text{Fig. 6.22 A point } P \text{ is imaged by the lens on point } T. \text{ How far apart have to be a point } P' \text{ so that the lens clearly image it at a point different than } T? \]

Let’s call \( t(PST) \) the traveling time of light to go from P to T passing through S.

- The condition that point P' focuses at a different point than T is that
  \( t(P'ST) \) and \( t(PST) \) have to be different.
  (otherwise they would focus at the same point.)
Again, how much different do \( t(P'ST) \) and \( t(PST) \) have to be so that we can conclude that these two paths do not belong to a set of rays forming a clear image?

- **The condition of resolution**
  Starting with the path PST and its corresponding time \( t(PST) \), let’s calculate the time \( t(P'ST) \) as the point \( P' \) moves out of axis away from \( P \).
  The condition of resolution (the ability to distinguish \( P \) and \( P' \)) states that
  
  “Two different point sources (\( P \) and \( P' \)) can be resolved only if one source (\( P \)) is imaged at such a point (\( T \)) that the time for the light rays from the other source (\( P' \)) to reach that point (\( T \)) differs by more than one period.”

In other words,

\[
 t(P'ST) - t(PST) > \text{one period of oscillation (of the radiation being used for imaging)} \\
 t(P'ST) - t(PST) > \frac{1}{\nu}
\]

Using the diagram in Fig. 6.21, and in terms of \( d \) and \( \theta \), the expression above becomes,

\[
 [ t(P'ST) - t(PST) ] = d \sin \theta / \nu = d \sin \theta / (c/n),
\]

where \( n \) is the index of refraction of the medium

**Fig. 17** Diagram to calculate the difference in travel time by two rays.
Thus, according to (28), the condition of resolution can be expressed as,
\[
d \sin \theta / (c/n) > \frac{1}{\nu}
\]
Or,
\[
d > \frac{\lambda}{n \sin \theta} \quad \text{Resolving power of a lens}
\]
Thus, $\frac{\lambda}{n \sin \theta}$ is the minimum distance that two points $P$ and $P'$ need to be separated in order to be imaged as two different points.

The quantity in the denominator is defined as the numeral aperture of the lens,
\[
NA \equiv n \sin \theta
\]
and the resolving power is typically expressed as
\[
\text{Resolving power} \ R = \frac{\lambda}{NA}
\]

**b Watching electrons through the microscope**

We wish to measure as accurate as possible the position of an electron. We have to take into account, however, that the very act of observing the electron with photons disturbs the electron’s motion.

The moment a photon interacts with an electron, it recoils in a way that cannot be completely determined (there will be an uncertainty in the possible ‘exact’ recoil direction.)

If we detect a signal through the microscope, it would because the photon of wavelength $\lambda$ (and of linear momentum $p=\hbar/\lambda$) has recoiled anywhere within the lens angle of view $2\theta'$. That is, the $x$-component of the photon ‘s momentum can be any value between $p\sin \theta$ and $-p\sin \theta$. $(\Delta p_x)_{\text{photon}} = 2p \sin \theta$. By conservation of linear momentum, the electron must have the same uncertainty in its $x$-component linear momentum $(\Delta p_x)_{\text{electron}} = (\Delta p_x)_{\text{photon}}$. Thus, the uncertainty in the $x$-component of the recoiled electron’s linear momentum is
\[
(\Delta p_x)_{\text{electron}} = 2p \sin \theta = 2(\hbar/\lambda) \sin \theta
\]
But, where is the electron (after the interaction with the photon)? How accurate can we determine its position. At the moment of detection of the recoiled photon the electron must have been somewhere inside the focused region of the lens. But the size of that region, according to expression (29), is not smaller than

$$\Delta x = \lambda / \sin \theta$$  \hspace{1cm} (33)

The product of the two uncertainties gives,

$$\Delta p_x \Delta x = 2h$$  \hspace{1cm} (34)

Fig. 18 The numerical aperture $n \sin \theta$ of the lens determines the uncertainty $\Delta p_x = 2p \sin \theta$ with which we can know the electron’s momentum.

QUESTION:

1. A light pulse of length duration ($\Delta t$); has wave components of energy within the range ($\Delta E$)₁.
The energy content can be revealed by making the pulse to pass through a spectrometer; let’s call this energy-content (or pulse fingerprint) “Spectrum-1.”

A similar pulse passes first through a slit. But the slit is opened only a fraction of time \((\Delta t)_2 = (\Delta t)_1/10\). (That is, an experimental attempt is made to reduce the temporal duration of the pulse.)

a) When the resulting pulse passes through a spectrometer, how different the new spectrum-2 would be compared to “Spectrum-1”?

b) Can we say that since the pulse is narrower it has gained components of a wider range of frequencies?

More bluntly, If a red color light-pulse passes through a slit that opens only an infinitesimal length of time, could we eventually be able to make this pulse to become a white-light pulse?

Answer:

a) Unless \((\Delta t)_2\) is not smaller than the grating’s \(1/\Delta v_{\text{min}}\) both spectrum would look the same.

Making the pulse shorter by narrowing the time the slit is open does not add new frequencies to the pulse.

b) No, a red-light pulse cannot be made a white-light pulse by this method.

Making the pulse narrower indeed widens its spectral Fourier components, but this has consequences on the uncertainty to the resolution with which we can determine such spectral response. Experimentally, if the pulse is made so narrow that \((\Delta t)_\text{pulse} < \text{grating’s } 1/\Delta v_{\text{min}}\) then the spectral line will be thicker. But this thickening is not because the pulse has more energies (in its spectral bandwidth); rather, it is because the inability of the grating to resolve better a spectral line.

In short: making the pulse shorter worsen the resolution to know is spectral content upon measuring, that is, increases the uncertainty in \(\Delta E\) (which is not the same to say that increases the \(\Delta E\) content.)
The spectrum produced by these two pulses will look alike. The spectral line will start looking much thicker.

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