Representation of traveling harmonic waves using PHASORS

\[
\text{wave} = f(x,t) = A \cos (kx - \omega t + \alpha)
\]

Let's call \( \theta = kx - \omega t + \alpha \)

\[
f(x,t) = A \cos (\theta)
\]

Notice, \( f(x,t) \) can be interpreted as the Real component of corresponding PHASOR \( z(\theta) \) as shown in the figure below.

This is a real wave.
We can detect it, we can measure it

\( z(\theta) \) is called a PHASOR

where \( \theta = kx - \omega t + \alpha \)

\( z(\theta) \) is a number in the complex plane.
Notice, as \( x \) and \( t \) change, the phasor rotates

Alternative notation of \( z(\theta) \): \( Ae^{i\theta} \) Euler's notation
Addition of waves using the method of phasors

Consider two waves, in general of different frequencies \((\omega_1, \omega_2)\) and different wave numbers \((k_1, k_2)\).

\[
\text{Wave 1: } f_1 = A_1 \cos (k_1 x - \omega_1 t + \phi_1) = A_1 \cos \theta_1 \\
\text{Wave 2: } f_2 = A_2 \cos (k_2 x - \omega_2 t + \phi_2) = A_2 \cos \theta_2
\]

At point \(P\), the waves add up.

We want to find \(f_1 + f_2\).

A convenient way to add waves is by the method of phasors.

\[
f_1(x,t) = A_1 \cos (\theta_1) = \text{Real component of } \{ z_1(\theta_1) \} \text{ phasor}
\]

\[
f_2(x,t) = A_2 \cos (\theta_2) = \text{Real component of } \{ z_2(\theta_2) \} \text{ phasor}
\]

\[
f_1(x,t) + f_2(x,t) = \text{Real component of } \{ z_1(\theta_1) + z_1(\theta_1) \}
\]

In different applications, it is easier to add up the phasors. Once the total sum (the phasor sum) is found, its horizontal component will be \(f_1 + f_2\).
PROCEDURE

STEP 1: Construct the individual corresponding phasors

\[ Z_1(\theta_1) \quad \text{or} \quad A_1 e^{i\theta_1} \quad \text{in Euler's notation} \]

and

\[ Z_2(\theta_2) \quad \text{or} \quad A_2 e^{i\theta_2} \quad \text{in Euler's notation} \]

Given,

\[ \theta_1 = k_1 x - \omega_1 t + \alpha_1 \]
\[ \theta_2 = k_2 x - \omega_2 t + \alpha_2 \]

\[ f_1 = A_1 \cos \theta_1 \]
\[ f_2 = A_2 \cos \theta_2 \]

We need to find \( z(\theta) \)
Notice: As the variables $x$ and $t$ vary, the phasors rotate

**STEP 2:** Work in the phasors world and evaluate

$$Z_1(\theta_1) + Z_2(\theta_2)$$

This sum will have the form

$$Z(\theta)$$  (see graph above)

(we have been told that it will be easier to find $Z(\theta)$)

**STEP 3:** $f_1 + f_2$ will be equal to

$$= \text{Real component of } \{ z(\theta) \}$$
Example: Consider the particular case in which the two added harmonic waves have the same frequency $\omega$ and the same wavelength $\lambda = 2\pi/k$

$$y_1 = A_1 \cos(kx - \omega t + \alpha_1) = A_1 \cos(\theta_1)$$
$$y_2 = A_2 \cos(kx - \omega t + \alpha_2) = A_2 \cos(\theta_2)$$

This implies constructing the corresponding phasors

Phasor-1: $z_1(\theta_1)$ where $\theta_1 = kx - \omega t + \alpha_1$

Phasor-2: $z_2(\theta_2)$ where $\theta_2 = kx - \omega t + \alpha_2$

It is clear, from the graph, that all the phasors rotate synchronously.
Notice, the graph above reduces to

Accordingly, the problem of adding the phasors

\[ z_1(\theta_1) \quad \text{where} \quad \theta_1 = kx - \omega t + \alpha_1 \]

and

\[ z_2(\theta_2) \quad \text{where} \quad \theta_2 = kx - \omega t + \alpha_2 \]

reduces to the addition of the simpler phasors

\[ z_1(\alpha_1) \quad \text{and} \quad z_2(\alpha_2) \]

Such a sum will have the form

\[ z(\alpha) = A e^{i\alpha} \]

where \( A \) and \( \alpha \) have to be determined in terms of \( A_1, \alpha_1, A_2 \) and \( \alpha_2 \).
Finding $A$ and $\alpha$

From the previous figure (re-drawn below)

$$A^2 = A_1^2 + A_2^2 + 2A_1 A_2 \cos(\alpha_2 - \alpha_1)$$

Similarly, also from the figure

$$A \sin \alpha = A_1 \sin \alpha_1 + A_2 \sin \alpha_2$$

$$A \cos \alpha = A_1 \cos \alpha_1 + A_2 \cos \alpha_2$$

$$\tan \alpha = \frac{A_1 \sin \alpha_1 + A_2 \sin \alpha_2}{A_1 \cos \alpha_1 + A_2 \cos \alpha_2}$$
**Example** Two waves travel along a string in the same direction

\[ f_1(x,t) = 4 \text{ mm} \cos(kx - wt) \]

\[ A_1 = 4 \text{ mm}, \quad \alpha_1 = 0 \]

\[ f_2(x,t) = 3 \text{ mm} \cos(kx - wt + \frac{\pi}{3}) \]

\[ A_2 = 3 \text{ mm}, \quad \alpha_2 = \frac{\pi}{3} \]

Find \( f_1(x,t) + f_2(x,t) \)

**Solution**

\( f_1(x,t) \rightarrow \) We build a corresponding phasor \( \vec{A}_1(0) \) (see graph.)

\( f_2(x,t) \rightarrow \) We build a corresponding phasor \( \vec{A}_2(\pi/3) \) (see graph.)

From the phasors graph

\[ A^2 = 16 + 9 + 2 \times 4 \times 3 \cos \frac{\pi}{3} = 37 \]

\[ \Rightarrow A = 6.1 \text{ mm} \]

\[ \tan \alpha = \frac{3 \sin \frac{\pi}{3}}{4 + 3 \cos \frac{\pi}{3}} = 0.47 \]

\[ \Rightarrow \alpha = 0.44 \text{ rad} \]
In summary, we have found,

\[ z_1(0) + z_2(\pi/3) = z(\alpha) \]

where \( A = 6.1 \text{ mm} \) and \( \alpha = 0.44 \text{ rad} \)

It means

\[ z_1(kx - \omega t + 0) + z_2(kx - \omega t + \pi/3) = z(kx - \omega t + \alpha) \]

Going back to the real world,

\[ f_1(x,t) + f_2(x,t) = \text{Horizontal component of } \{ z(kx - \omega t + \alpha) \} \]

\[ = 6.1 \text{ mm } \cos(kx - \omega t + 0.44 \text{ rad}) \]
\[ = 6.1 \text{mm} \cos (kx - \omega t + 0.44 \text{rad}) \]

Notice:
The phase 0.44 rad of the resulting wave is between the phase values of the component waves.

\[ 0 < 0.44 \text{rad} < \frac{\pi}{3} \]
EXAMPLE  Addition of two waves having a phase difference $\delta$.

\[ E_1 = E_{10} \cos kx \quad (\alpha_1 = 0) \]
\[ E_2 = E_{20} \cos (kx + \delta) \]

$\delta$ is given

Even though

\[ z_1 = E_{10} e^{ikx} \]
\[ z_2 = E_{20} e^{i(kx + \delta)} \]

it is enough to consider the addition of

\[ E_{10} + E_{20} e^{i\delta} \]

The sum will be written as $E_0 e^{ikx}$.

Solution:

\[ E_0^2 = E_{10}^2 + E_{20}^2 + 2E_{10}E_{20} \cos \delta \]

\[ \tan \alpha = \frac{E_{20} \sin \delta}{E_{10} + E_{20} \cos \delta} \]

\[ z_1 + z_2 = [E_0 e^{i\alpha}] e^{i(kx - \omega t)} = E_0 e^{i(kx - \omega t + \alpha)} \]

then

\[ E_0 \cos kx + E_{20} \cos (kx + \delta) = E_0 \cos (kx - \omega t + \alpha) \]
Phasors Method

Generalization to add many waves

\[ y = 3 \cos \left( wt + \frac{\pi}{6} \right) + \]
\[ 4 \cos \left( wt + \frac{\pi}{2} \right) + \]
\[ 2 \cos \left( wt - \pi \right) + \]
\[ 2.5 \cos \left( wt + \frac{3}{4} \pi \right) \]

Since all the waves have the same frequency, we set to add

\[ 3e^{i\pi/6} + 4e^{i\pi/2} + 2e^{i\pi} + 2.5 e^{i3\pi/4} = \]
\[ = Ae^{i\alpha} \]

Thus

\[ y = A \cos \left( wt + \alpha \right) \]
An analytical expression for $A$, in terms of the amplitudes $A_i$ of the component waves, can be obtained if we consider the phasors $A_i e^{i\alpha_i}$ as if they were vectors.

$$|\vec{A}|^2 = \vec{A} \cdot \vec{A} = (\sum_i A_i^2) \cdot (\sum_i \vec{A}_i)$$

$$A^2 = \sum_i A_i^2 + \sum_i \sum_{j \neq i} A_i A_j \cos (\alpha_i - \alpha_j)$$

and

$$\tan \alpha = \frac{\sum_i A_i \sin \alpha_i}{\sum_i A_i \cos \alpha_i}$$