2.2.C Analogy between
- “electronic excitations in an atom” and the
- “mechanical motion of a forced harmonic oscillator”

How to choose the value of the corresponding spring constant $k$?

Resonant Absorption

Let's assume we know the particular transition energy $\Delta E$. Therefore we know $\omega_o$, also called $\omega_{res}$.

Mechanical resonance

We identify the mechanical resonance frequency with the optical resonance frequency $\omega_o$.

$$w_{rsc} = \sqrt{\frac{k}{m_e}}$$

$$k \equiv m_e \omega_o^2$$
How to choose the value of the damping constant "b"?

If $\omega_f = \omega_0$ the incident photon is absorbed by the atom.

Resonant Absorption

Light scattering

Charged mechanical oscillator

Input energy

Atom driven by external radiation of frequency $\omega_f$
The emission of electromagnetic waves by the accelerated charge constitutes the channel of dissipative energy.

The specific value of the electromagnetic damping parameter "b" (in terms of the atomic physical constants) is provided in very much detail in section IV of these Applied Optics notes. Here, in the next section, we provide a summarized version.
Electromagnetic radiation damping
Modeling optical excitation with a simple harmonic oscillator

1. Emission of radiation by an accelerated charge
2. Emission of radiation by charge undergoing harmonic oscillations
3. Modeling optical excitation with a simple harmonic oscillator

Ref: Feynman Lectures Vol-1, Chapter 32

1. Emission of radiation by an accelerated charge

Electromagnetic radiation is emitted by charges in accelerated motion.

![Diagram of an accelerated charge emitting radiation]

The electric field $E$, at a distance $r$ from the charge, and at an angle $\theta$ from the axis of the charge motion, is perpendicular to the line of sight.

**Fig. 1** A charge $q_e$ that moves with acceleration $a$ emits electromagnetic radiation.

$$E(r,t) = -\frac{q_e \sin(\theta)}{4\pi \varepsilon_0 c^2} \frac{1}{r} a(t - r/c)$$  

where $a$ is the acceleration of the particle,

$\theta$ the angle between the vector acceleration $a$ and the vector position $r$.

$c$ is the speed of light
Traveling electromagnetic energy: The Poynting vector $\vec{S}$

$$\vec{S} = \varepsilon_o E^2 \vec{c}$$  

Its magnitude $S$ gives the amount of energy per square meter per unit time that passes through a surface that is normal to the direction of propagation.

Radiation power $P$ emitted by an accelerated charge

![Diagram](image)

$$P = \int S \, dA = \int [\varepsilon_o c^2 q_e^2 \sin^2(\theta) \frac{1}{r^2} a'^2] \, dA$$

$$P = \frac{q_e^2}{6\pi\varepsilon_o c^3} a'^2$$

where $a' = a(t') = a(t - r/c)$

2. Emission of radiation by charge undergoing harmonic oscillations

Consider a charge $q_e$ attached to a mechanical spring that is oscillating with angular frequency $\omega$

$$x(t) = \text{Real} \{ x_o e^{j\omega t} \} = x_o \cos(\omega t)$$  \hspace{1cm} (5)

Complex variable

Here, $\omega$ is not necessarily the natural frequency $\omega_0$ of the
mechanical oscillator (see Fig 2). For example, the oscillator may be being driven by an external source of arbitrary frequency \( \omega \).

![Fig. 3 Charge attached to a spring of appropriate natural frequency \( \omega_o \).](image)

In what follows we will be evaluating time averages of physical quantities (i.e. \( f \)) over one period \( T = 2\pi / \omega \) of oscillation:

\[
< f > = \frac{1}{T} \int_0^T f(t) \, dt
\]  

(6)

Notice, for example, that the time average of \( \cos^2(\omega t) \) over one period is \( \frac{1}{2} \).

We also will be using indistinctly \( \cos(\omega t) \) or \( e^{j\omega t} \). For the latter it is assumed that we will take the real part after the calculations in order to obtain a proper (classical) physical interpretation.

From expression (5) we calculate an acceleration,

\[
a' = -\omega^2 x_o e^{j\omega t}
\]  

(7)

Whose average is,

\[
< a' >^2 = \left( \frac{1}{2} \right) \omega^4 |x_o|^2.
\]  

(8)
Replacing these values in expression (4) gives,

\[
\langle P \rangle = \frac{q_e^2 |x_o|^2}{12\pi\varepsilon_o c^3} \omega^4
\]

Total average power emitted by a charge \(q_e\) oscillating with amplitude \(|x_o|\).

where \(x_o\) and \(\omega\) are the amplitude and angular frequency of the charge’s oscillations, respectively. Notice, the larger the amplitude \(|x_o|\), the higher the power emitted by the charge.

3. Electromagnetic radiation damping

To calculate the amplitude of oscillation \(|x_o|\), let’s model the atomic excitation with a forced harmonic oscillator.

Consider the equation of motion for the charge \(q_e\), attached to a mechanically frictionless spring,

\[
m \frac{d^2x}{dt^2} + m\gamma \frac{dx}{dt} + kx = q_e E_o e^{i\omega t}
\]  (10)

Here the term

\[
m\gamma \frac{dx}{dt}
\]  (11)

accounts for the dissipation energy due to the emission of electromagnetic radiation by the accelerated charge. (The term is chosen proportionally to the velocity just to facilitate the solution of the equation (10)).

A stationary solution of (11) is given by (as demonstrated in the next section),

\[
x = [x_o e^{j\varphi}] e^{j\omega t}
\]  (12)

where
\[ x_o = x_o(\omega) = \frac{(q_e / m_o)E_o}{\left[(\omega_o^2 - \omega^2)^2 + \gamma^2 \omega^2\right]^{1/2}} \]  \hspace{1cm} (13)

and

\[ \varphi = \tan^{-1}\left(-\frac{\gamma \omega}{\omega_o^2 - \omega^2}\right) \]  \hspace{1cm} (14)

Expression (13) indicates that the amplitude of oscillation \( x_o \) (and hence the acceleration) of the charge depends on the external incident radiation's frequency.

**Comparison between expressions (9) and (11)**

If expression (10) is going to be used to model the optical excitations of an atom,

the term \( m\gamma \frac{dx}{dt} \) in (10), that characterizes the dissipation of energy of the mechanical oscillator,

should be compatible with

expression (9), \( <P> = \frac{q_e^2 |x_o|^2}{12\pi e_o c^3 \omega^4} \), that gives the energy dissipated by the oscillator.

How to express mathematically this compatibility?

**Strategy:** Find the dissipation power associated to the term \( m\gamma \frac{dx}{dt} \).

Once found, it should be equal to power \( <P> = \frac{q_e^2 |x_o|^2}{12\pi e_o c^3 \omega^4} \).

The power dissipate by a mechanical oscillator, modelled by (10), is given by,
Force $\times$ velocity = $[m \gamma \frac{dx}{dt}](\frac{dx}{dt})$

using (12)

$= [m \gamma (j \omega x)]((j \omega x)) = -m \gamma \omega^2 x^2$.

The magnitude of the average of $\text{force} \times \text{velocity}$ will be

$< \text{Force} \times \text{velocity} > = -(1/2)m \gamma \omega^2 x_o^2$ \hspace{1cm} \text{(15)}$

This is the expression that must be compatible with expression (9). Accordingly, if the mechanical oscillator model (10) is to be used to describe the optical excitation, it must occur that,

$-(1/2)m \gamma \omega^2 x_o^2 = <P> = \frac{q^2 x_o^2}{12 \pi \varepsilon_o c^3} \omega^4$

This allows to identify $\gamma = \frac{q^2}{6 \pi \varepsilon_o m c^2} \omega^2$ or, rearranging terms,

$\gamma = \frac{2}{3c} \frac{q^2}{4 \pi \varepsilon_o m c^2} \omega^2$

\hspace{1cm} \text{electromagnetic radiation damping} \hspace{1cm} \text{(16)}$

$m \frac{d^2x}{dt^2} + m \gamma \frac{dx}{dt} + kx = q_e E_o e^{j \omega t}$

$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = q_e E_o e^{j \omega t}$
Light-matter interaction will therefore be modelled by a harmonic forced damped oscillator.

\[ m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F_0 \cos(\omega_f t) \]

- \( m \): electron's mass
- \( x \): is the "position" of the electron
- \( b = b(\omega_f, m, ...) \): related to the rate at which the accelerated electron re-emit light.
- \( k = m \omega_0^2 \): one of the atom's disscast resonant absorption frequencies.
- \( F_0 = eE_0 \): external electric field amplitude.
- \( \omega_f \): driving frequency
- \( \omega_0 \): resonance frequency
- \( k \): spring constant
- \( b \): damping coefficient

Light

Matter
Solving the forced harmonic oscillator equation, Eq. 1, in complex variable

First, we apply a trick:

Let \( y = y(t) \) be the solution of

\[
 m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + ky = F_0 \sin(\omega_f t) \tag{2}
\]

Multiply (2) by \( i \)

\[
 m \frac{d^2 (iy)}{dt^2} + b \frac{diy}{dt} + k(iy) = F_0 [i \sin(\omega_f t)] \tag{3}
\]

(1) + (3) gives

\[
 m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + k z = F_0 e^{i \omega_f t} \tag{4}
\]

where \( z(t) = x(t) + iy(t) \)

If we find a solution in (4), let's say \( z(t) \)
then a solution to Eq (1) can be obtained by taking \( x(t) = \text{Real}\{z(t)\} \)
Solving Eq 4

Since the driving force is harmonic of frequency \( w_f \), we guess the displacement \( z(t) \) will also be harmonic of frequency \( w_f \). Potentially, there may be phase difference between \( F_0 \exp(iw_f t) \) and \( z(t) \).

Thus, we propose a solution of the form,

\[
z(t) = A \exp(i(w_f t + \phi))
\]

where \( A \) and \( \phi \) can depend on \( w_f \) and other parameters.

\[\text{Eq} \ 5\]

\[\text{in Eq} \ 4\) gives,

\[
(-w_f^2 + w_0^2 + i \frac{b}{m} w_f) A \exp(i \phi) = \frac{F_0}{m}
\]

\[\Rightarrow \]

\[
A \exp(i \phi) = \frac{1}{\frac{(-w_f^2 + w_0^2 + i \frac{b}{m} w_f)}{m}} \frac{F_0}{m}
\]

\[
= \frac{(-w_f^2 + w_0^2) - i \frac{b}{m} w_f}{(-w_f^2 + w_0^2)^2 + \left( \frac{b}{m} w_f \right)^2} \frac{F_0}{m}
\]
Notice, the complex number in the numerator can be expressed as,

\[
(-\omega_f^2 + \omega_o^2) - i \frac{b}{m} \omega_f = \left[(-\omega_f^2 + \omega_o^2)^2 + \left(\frac{b}{m} \omega_f\right)^2\right]^{\frac{1}{2}} e^{i \tan^{-1} \frac{-\frac{b}{m} \omega_f}{\omega_o^2 - \omega_f^2}}
\]

\[\Rightarrow\]

\[A e^{i \theta} = \frac{F_o/m}{\left[(-\omega_f^2 + \omega_o^2)^2 + \left(\frac{b}{m} \omega_f\right)^2\right]^{\frac{1}{2}}} e^{i \tan^{-1} \frac{-\frac{b}{m} \omega_f}{\omega_o^2 - \omega_f^2}}
\]

\[A(\omega_f) = \frac{F_o/m}{\left[(-\omega_f^2 + \omega_o^2)^2 + \left(\frac{b}{m} \omega_f\right)^2\right]^{\frac{1}{2}}}
\]

\[\theta(\omega_f) = \tan^{-1} \frac{-\frac{b}{m} \omega_f}{-\omega_f^2 + \omega_o^2}
\]
Summary.

\[ m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + k z = F_0 e^{i \omega_f t} \]

This equation admits solutions of the form

\[ z(t) = A e^{i (\omega_f t + \phi)} \]

where \( A = A(\omega_f) \) and \( \phi = \phi(\omega_f) \) are given in expression (5).

Accordingly,

The solution to \( \mathcal{E} \)

\[ m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + k x = F_0 \cos(\omega_f t) \]

is given by

\[ x(t) = \text{Re} \{z(t)\} \]

\[ = A(\omega_f) \cos(\omega_f t + \phi) \]
Graphic analysis of the solutions (phasors)

Velocity phasor \( z \)

- Phasor force: \( F_0 e^{i \omega_f t} \)
- Phasor position: \( A e^{i(\omega_f t + \phi)} \)
- Phasor velocity: \( A \omega_f e^{i(\omega_f t + \phi + \pi/2)} \)

\[
\frac{dz}{dt} = \frac{d}{dt} A e^{i(\omega_f t + \phi)} = i \omega_f A e^{i(\omega_f t + \phi)}
\]

since \( i = e^{i(\pi/2)} \)

\[
= \omega_f A e^{i(\omega_f t + \phi + \pi/2)}
\]

Notice, since \( \phi \) is always negative, the POSITION PHASOR always lags the FORCE PHASOR
Variation of $\phi$

$\omega_f$ variable driving frequency

$\omega_0$ oscillator natural frequency (fixed value)

At low $\omega_f$

$\frac{(-\omega_f^2 + \omega_0^2)}{2} \rightarrow -\frac{b}{m} \omega_f$

At higher $\omega_f$ but $\omega_f < \omega_0$

$\omega_f \approx \omega_0$

$\phi \approx 90^\circ$

At $\omega_f > \omega_0$

$\frac{(-\omega_f^2 + \omega_0^2)}{2} \rightarrow \phi$
$A_{\text{max}} = \frac{F_0}{\kappa} \left( \frac{1}{Q} \right) \left( 1 - \frac{1}{4Q^2} \right)^{1/2}$

For $Q \gg 1$

$A_{\text{max}} \propto Q \frac{F_0}{\kappa}$

\[ \omega_0' = \sqrt{\omega_0^2 - \frac{1}{2} \left( \frac{b}{m} \right)^2} \]

\[ = \omega_0 \sqrt{1 - \frac{1}{2Q^2}} \quad (Q \approx \frac{m}{b} \omega_0) \]

$\omega_0' \approx \omega_0$ for high $Q$