First, let's recall

\[ \frac{1}{\sqrt{\varepsilon \mu}} = \nu = \text{speed of light propagation in a material} \]

\[ \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = c = \text{speed of light propagation in vacuum} \]

We call this ratio

\[ \sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}} = \frac{c}{\nu} = "\text{index of refraction}" \]
The Maxwell Eqs. lead to the following equation to be satisfied by each component of the electric field and magnetic field waves:

\[
\frac{\partial^2 E_y}{\partial x^2} - \frac{j}{\left(\frac{1}{\varepsilon \mu}\right)} \frac{\partial^2 E_y}{\partial t^2} = 0
\]

(1)

where we identify the propagation velocity,

\[v = \frac{j}{\sqrt{\varepsilon \mu}}\]

\[\varepsilon \text{ and } \mu \text{ characterize the medium in which the wave propagates}\]

The equation above admits solutions of the form,

\[E_y = f(z - vt)\]

only this part is important

this can be any arbitrary function.

More general,

\[E_y = f(z - vt) + g(z + vt)\]

profile advancing in the \(X^+\) direction

profile advancing in the \(X^-\) direction
In particular, harmonic functions of the form,

\[ E_y = E_m \cos (kx - \omega t) \]  \hspace{1cm} (2)

also satisfy Eq. (1), provided that,

\[ \frac{\omega}{k} = v \]  \hspace{1cm} \text{(where } v^2 = \frac{c^2}{(\varepsilon / \varepsilon_0)} \text{)}

Thus, expression (2) can also be written as,

\[ E_y = E_m \cos \left( \frac{\omega}{v} x - \omega t \right) \]

Notice also,
If \( \vec{E}_x \) and \( \vec{E}_y \) are considered independent of each other then we could make the initially linearly polarized light

\[
\vec{E} = E_x \hat{i} + E_y \hat{j}
\]

to pass through a thin piece of special material that has different indices of refraction along its different axis. That is, a material with \( n_x \neq n_y \). According to expression (3), \( E_x \) and \( E_y \) would, consequently, undergo different phase shifts.
Light travels at different speeds, depending on the state of polarization.
CASE: Light linearly polarized along the x-axis

\[ \mathbf{v}_1 = \frac{c}{n_x} \]

\[ \mathbf{E} \] travels inside the material with velocity \( \mathbf{v}_1 = \frac{c}{n_x} \)

CASE: Light linearly polarized along the Y-axis

\[ \mathbf{v}_2 = \frac{c}{n_y} \]

\[ \mathbf{E} \] travels inside the material with velocity \( \mathbf{v}_2 = \frac{c}{n_y} \)
CASE: Light linearly polarized in the XY-plane, and along a line that makes 45 degrees with the X-axis.

Let's assume $n_x < n_y$.

In this region $E_x$ and $E_y$ are in phase.

In this region $E_x$ and $E_y$ are not in phase.

The two figures above show the spatial variation of the fields $E_x$ and $E_y$ at a given instant of time $t = t_0$. 

$n_y > n_x$ implies $\lambda_x < \lambda_y$. 

$n_x$ 

$n_y$
Now, let's describe mathematically the waves in the different regions, outside and inside the birefringent material.

For $z < 0$

\[
E_x = E_m \cos \left( \frac{w}{c} n \frac{z}{2} - \omega t \right) \\
E_y = E_m \cos \left( \frac{w}{c} n \frac{z}{2} - \omega t \right)
\]

Incident light is linearly polarized in the XY-plane, and along a line that makes 45 degrees with the X-axis.

For $0 < z < d$

\[
E_x = E_t \cos \left( \frac{w}{c} n_x z - \omega t \right) \\
E_y = E_t \cos \left( \frac{w}{c} n_y z - \omega t \right)
\]

Phase accumulated from $z=0$ to an arbitrary value of $z$ (where $z<d$)
Incident light is linearly polarized in the XY-plane, and along a line that makes 45 degrees with the X-axis.

For $0 < z < d$

**Components $E_x$ and $E_y$**

**ENTER in PHASE**

At $z=0$

$$E_x = E_m \cos \left( \omega n_x z - \omega t \right)$$

$$E_y = E_m \cos \left( \omega n_y z - \omega t \right)$$

At $z=d$

Extra phase accumulated from $z=0$ to $z=d$

$$E_x = E_m \cos \left( \frac{\omega}{c} n_x d - \omega t \right)$$

$$E_y = E_m \cos \left( \frac{\omega}{c} n_y d - \omega t \right)$$

In general, the components $E_x$ and $E_y$ COME OUT from the birefringent material OUT OF PHASE

**$E_x$ and $E_y$ are**

- **in PHASE**

- **out of phase**
How to express the fields in the region $z>0$

For $z>d$

To find an expression for the fields in the region $z>d$, all we have to do is to account for the accumulated change of phase from $z=0$ to an arbitrary $z$ ($z>d$). Obviously, we need to take into account the phase change through the slab (where the index of refraction is $n_x$ or $n_y$), plus the phase-change from $z=d$ to an arbitrary $z$ (where the index of refraction is $n$ for both)

$$E_x(z,t) = E_m \cos \left[ (\omega/c)n_x d + (\omega/c)n(z-d) - \omega t \right]$$

$$E_y(z,t) = E_m \cos \left[ (\omega/c)n_y d + (\omega/c)n(z-d) - \omega t \right]$$

Expressions for the fields in the region $z>d$

Incident light is linearly polarized in the XY-plane, and along a line that makes 45 degrees with the X-axis.
\[ \eta_x \neq \eta_y \]

Inside the material:

In the region \( z > d \)

\[ E_x = E_m \cos \left( \frac{w}{c} \eta (z-d) - \omega t + \delta_1 \right) \]

\[ E_y = E_m \cos \left( \frac{w}{c} \eta (z-d) - \omega t + \delta_2 \right) \]

In the region \( z < 0 \)

\[ E_x = E_m \cos \left( \frac{w}{c} \eta x - \omega t \right) \]

\[ E_y = E_m \cos \left( \frac{w}{c} \eta y - \omega t \right) \]

Where

\[ \delta_1 = \frac{w}{c} \eta_x d = k_x d \]

\[ \delta_2 = \frac{w}{c} \eta_y d = k_y d \]
Notice, the phase difference between the electric field components $E_x$ and $E_y$ at $z = d$ (when the electromagnetic waves come out of the birefringent material) is

$$\Delta \phi = \frac{\omega}{c} (n_y - n_x) d = (k_y - k_x) d$$

Example: Quarter-wave plate

If $\Delta \phi = \pm \frac{\pi}{2}$, then the electric field vector

$$\vec{E} = E_x \hat{x} + E_y \hat{y}$$

will be rotating either clockwise (-) or counter clockwise (+).

$$\Delta \phi = \frac{\omega}{c} (n_y - n_x) d = \pm \frac{\pi}{2}$$

$$\left(n_y - n_x\right) d = \pm \frac{\omega}{c} \frac{\pi}{2} = \pm \frac{c}{2\pi} \frac{\pi}{2} = \pm \frac{\lambda_0}{4}$$

Condition on the thickness “d” of the birefringent material to convert incident linearly polarized light into circularly polarized
Principal indices of refraction of several crystals

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<th>$n_x$</th>
<th>$n_y$</th>
<th>$n_z$</th>
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<td>1.6583</td>
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</table>

Birefringent material
Index of refraction $n_x$, $n_y$

$(n_y - n_x)d = \frac{\lambda}{4}$

Left circularly polarized light
Right circularly polarized light

Linearly polarized light
Circularly polarized light
Example

For birefringent crystals, what is the minimum thickness $d$ to convert linearly polarized light of $\lambda = 500 \text{ nm}$ into circularly polarized?

\[
\frac{\Delta n}{|n_y - n_x|} = 1.6583 - 1.4864 = 0.1719
\]

\[
d = \frac{500 \text{ nm}}{4 \times 0.1719} = 0.73 \mu\text{m}
\]

Calcite

\[
\frac{\Delta n}{|n_y - n_x|} = 1.5533 - 1.5442 = 0.0091
\]

\[
d = \frac{500 \text{ nm}}{4 \times 0.0091} = 13.7 \mu\text{m}
\]

Quartz
Summary

Light is left-circularly polarized if $k_o d - k_e d = \pi/2$

Light is right-circularly polarized if $k_o d - k_e d = -\pi/2$