First, let's recall

\[ \frac{1}{\sqrt{\varepsilon \mu}} = V = \text{speed of light propagation in a material} \]

\[ \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = c = \text{speed of light propagation in vacuum} \]

We call this ratio

\[ \sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}} = \frac{c}{V} = \text{"index of refraction"} \]
The Maxwell Eqs. lead to the following equation to be satisfied by each component of the electric field and magnetic field waves:

\[
\frac{\partial^2 E_y}{\partial z^2} - \frac{1}{\sqrt{\varepsilon \mu}} \frac{\partial^2 E_y}{\partial t^2} = 0
\]  

(1)

where we identify the propagation velocity,

\[ V = \frac{1}{\sqrt{\varepsilon \mu}} \]

The equation above admits solutions of the form,

\[ E_y = f(z - Vt) \]

More general,

\[ E_y = f(z - Vt) + g(z + Vt) \]

only this part is important

this can be any arbitrary function.
In particular, harmonic functions of the form,

\[ E_y = E_m \cos(kx - \omega t) \]  

also satisfy Eq. (1), provided that,

\[ \frac{\omega}{k} = v \quad (where \quad v = \frac{1}{\nu}) \]

Thus, expression (2) can also be written as,

\[ E_y = E_m \cos\left(\frac{\omega}{v} x - \omega t\right) \]

Notice also,
If \( \vec{E}_x \) and \( \vec{E}_y \) are considered independent of each other then we could make the initially linearly polarized light

\[
\vec{E} = \vec{E}_x \hat{i} + \vec{E}_y \hat{j}
\]

to pass through a thin piece of special material that has different indices of refraction along its different axis. That is, a material with \( n_x \neq n_y \).

According to expression (3), \( E_x \) and \( E_y \) would, consequently, undergo different phase shifts.
Light travels at different speeds, depending on the state of polarization.

These expressions describe the fields $E_x$ and $E_y$ inside the birefringent material:

$$E_x = E_m \cos \left( \frac{\omega}{c} \eta_x z - \omega t \right)$$

$$E_y = E_m \cos \left( \frac{\omega}{c} \eta_y z - \omega t \right)$$
CASE: Light linearly polarized along the x-axis

\[ v_1 = \frac{c}{n_x} \]

\[ \vec{E} \] travels inside the material with velocity \( v_1 = \frac{c}{n_x} \)

CASE: Light linearly polarized along the Y-axis

\[ v_2 = \frac{c}{n_y} \]

\[ \vec{E} \] travels inside the material with velocity \( v_2 = \frac{c}{n_y} \)
CASE: Light linearly polarized in the XY-plane, and along a line that makes 45 degrees with the X-axis

Let's assume $n_x < n_y$

In this region, $E_x$ and $E_y$ are in phase.

$\eta_y > \eta_x$ implies $\lambda_x < \lambda_y$

The two figures above show the spatial variation of the fields $E_x$ and $E_y$ at a given instant of time $t = t_0$. 
Now, let's describe mathematically the waves in the different regions, outside and inside the birefringent material.

For $z < 0$

\[
E_x = E_m \cos \left( \frac{w}{c} n z - \omega t \right) \\
E_y = E_m \cos \left( \frac{w}{c} n z - \omega t \right)
\]

For $0 < z < d$

\[
E_x = E_t \cos \left( \frac{w}{c} n_x z - \omega t \right) \\
E_y = E_t \cos \left( \frac{w}{c} n_y z - \omega t \right)
\]

Phase accumulated from $z=0$ to an arbitrary value of $z$ (where $z<d$)
Let's evaluate the phase-change that each wave undergoes when crossing the birefringent slab

For $z < 0$

The fields vary as a function of time like,

\[ E_x = E_t \cos(-\omega t) \]

\[ E_y = E_t \cos(-\omega t) \]

For $0 < z < d$

The fields vary as a function of time like,

\[ E_x = E_t \cos\left(\frac{\omega}{c} n_x d - \omega t\right) \]

\[ E_y = E_t \cos\left(\frac{\omega}{c} n_y d - \omega t\right) \]

Phase accumulated from $z=0$ to $z=d$
In general, the components $E_x$ and $E_y$ come out from the birefringent material out of phase.

$E_x = E_0 \cos(\omega z - \omega t)$

$E_y = E_0 \cos(\omega z - \omega t)$

$E_z = 0$

$E = E_0 \cos(\omega z - \omega t)$

$E_x = E_0 \cos(\omega z - \omega t)$

$E_y = E_0 \cos(\omega z - \omega t)$

Components enter in phase.
How to express the fields in the region $z>0$

For $z >d$

To find an expression for the fields in the region $z>d$, all we have to do is to account for the accumulated change of phase from $z=0$ to an arbitrary $z$ ($z>d$). Obviously, we need to take into account the phase change through the slab (where the index of refraction is $n_x$ or $n_y$), plus the phase-change from $z=d$ to an arbitrary $z$ (where the index of refraction is $n$ for both)

$$E_x(z,t) = E_m \cos \left[ (\omega/c)n_x d + (\omega/c)n(z-d) - \omega t \right]$$

$$E_y(z,t) = E_m \cos \left[ (\omega/c)n_y d + (\omega/c)n(z-d) - \omega t \right]$$

- Phase accumulated from $z=0$ to $z=d$ (across the slab)
- Phase accumulated from $z=d$ to $z$ (across the air)
\( \eta_x \neq \eta_y \) inside the material

In the region \( z > d \)

\[
E_x = E_m \cos \left( \frac{w}{c} \eta (z-d) - wt + \delta_x \right)
\]

\[
E_y = E_m \cos \left( \frac{w}{c} \eta (z-d) - wt + \delta_y \right)
\]

where

\[
\delta_1 = \frac{w}{c} \eta_x d = k_x \cdot d
\]

\[
\delta_2 = \frac{w}{c} \eta_y d = k_y \cdot d
\]

In the region \( z < 0 \)

\[
E_x = E_m \cos \left( \frac{w}{c} \eta z - wt \right)
\]

\[
E_y = E_m \cos \left( \frac{w}{c} \eta z - wt \right)
\]
Notice, the phase difference between the electric field components $E_x$ and $E_y$ at $z = d$ (when the electromagnetic waves come out of the birefringent material) is

$$\Delta \phi = \frac{\omega}{c} (n_y - n_x) d = (\gamma_y - \gamma_x) d$$

**Example: Quarter-wave plate**

If $\Delta \phi = \pm \frac{\pi}{2}$, then the electric field vector $\vec{E} = E_x \hat{x} + E_y \hat{y}$ will be rotating either clockwise (−) or counterclockwise (+).

$$\Delta \phi = \frac{\omega}{c} (n_y - n_x) d = \pm \frac{\pi}{2}$$

$$(n_y - n_x) d = \pm \frac{\lambda_0}{2\pi} = \pm \frac{\frac{\pi}{2}}{2\pi} = \frac{\lambda_0 \pi}{2\pi}$$

Condition on the thickness of the birefringent material to convert incident linearly polarized light into circularly polarized.
Principal indices of refraction of several crystals

<table>
<thead>
<tr>
<th>Material</th>
<th>$n_x$</th>
<th>$n_y$</th>
<th>$n_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calcite</td>
<td>1.4864</td>
<td>1.6583</td>
<td></td>
</tr>
<tr>
<td>Quartz</td>
<td>1.5533</td>
<td>1.5442</td>
<td></td>
</tr>
<tr>
<td>Mica</td>
<td>1.5692</td>
<td>1.6049</td>
<td>1.6117</td>
</tr>
<tr>
<td>Topaz</td>
<td>1.6155</td>
<td>1.6181</td>
<td>1.6250</td>
</tr>
</tbody>
</table>

- **Linearly polarized**
- **Birefringent material**
- Index of refraction $n_x, n_y$
- $(n_y - n_x)d = \frac{\lambda}{4}$

Left - circularly polarized light
Right - circularly polarized light
Example: For birefringent crystals, what is the minimum thickness $d$ to convert linearly polarized light of $\lambda = 500\,\text{nm}$ into circularly polarized?

**Calcite**

$$\frac{\nu_y - \nu_x}{4} = 1.6583 - 1.4864 = 0.1719$$

$$d = \frac{500\,\text{nm}}{4 \times 0.1719} = 0.73\,\mu\text{m}$$

**Quartz**

$$\frac{\nu_y - \nu_x}{4} = 1.5533 - 1.5442 = 0.0091$$

$$d = \frac{500\,\text{nm}}{4 \times 0.0091} = 13.7\,\mu\text{m}$$
Summary

If \(k_o \cdot d - k_e \cdot d = \pi/2\)
Light is left-circularly polarized

If \(k_o \cdot d - k_e \cdot d = -\pi/2\)
Light is right-circularly polarized

\(E_y(z,t) = E_m \cos [k_o \cdot d + k \cdot (z-d) - \omega t]\)

\(E_x(z,t) = E_m \cos [k_e \cdot d + k \cdot (z-d) - \omega t]\)