Propagation of plane waves in an anisotropic material

For the case of a dielectric material, in the absence of free charges, the ME have the form

\[
\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = 0 \quad \nabla \cdot \mathbf{B} = 0
\]

\[
\nabla \times \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} = 0 \quad \varepsilon_0 c^2 \nabla \times \mathbf{B} = \frac{\partial}{\partial t} (\varepsilon_0 \mathbf{E} + \mathbf{P})
\]

(1)

Here \( \mathbf{P} \) is the polarizability vector, which accounts for the response of the bound charges to the presence of the electromagnetic wave.

Equation (1) is also typically expressed in terms of the electric displacement vector \( \mathbf{D} \)

\[
\mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P}
\]

(2)

\[
\nabla \cdot \mathbf{D} = 0 \quad \nabla \cdot \mathbf{B} = 0
\]

(3)

In a non-isotropic material the expression \( \mathbf{D} = \varepsilon \mathbf{E} \) no longer holds, since \( \mathbf{P} \) is, in general, no parallel to \( \mathbf{E} \).

When \( \mathbf{D} = \varepsilon \mathbf{E} \) we obtained plane waves \( \mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \) as solutions of the ME.

This time, when \( \mathbf{D} \) and \( \mathbf{E} \) are no longer parallel, we will still attempt pursuing plane waves as solutions of the ME.

\[
\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}
\]

\[
\mathbf{D}(\mathbf{r}, t) = \mathbf{D}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}
\]

\[
\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}
\]

(4)

where \( \mathbf{k} \) is a wave-vector to be determined.

Replacing (4) in (3), the first ME \( \nabla \cdot \mathbf{D} = 0 \) implies,

\[
\mathbf{k} \cdot \mathbf{D} = 0
\]

(6)

The second ME \( \nabla \cdot \mathbf{B} = 0 \) implies,
\[ \vec{k} \cdot \vec{B} = 0 \]  

Hence, \( \vec{D} \) and \( \vec{B} \) are perpendicular to the wave-vector \( \vec{k} \).

However, the latter does not apply to \( \vec{E} \)

\[ \vec{k} \cdot \vec{E} = ? \]

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**The Fresnel’s ellipsoid**

The fourth ME \( \varepsilon_0 c^2 \nabla \times \vec{B} = \frac{\partial}{\partial t} \vec{D} \) implies,

\[\varepsilon_0 c^2 \vec{k} \times \vec{B} = -\omega \vec{D}\]  

(8)

Since \( \vec{D} \) is perpendicular to \( \vec{k} \) and \( \vec{B} \), one can conveniently choose the axis such that,

\[ \vec{B} \]

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The third ME \( \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{B} = 0 \) implies \( i\vec{k} \times \vec{E} - i\omega \vec{B} = 0 \), or

\[ \vec{B} = \frac{1}{\omega} \vec{k} \times \vec{E} \]  

(9)
This expression, together with the previous graph, implies that \( \vec{E} \) should lie on the plane defined by \( \vec{k} \) and \( \vec{D} \).

\[ \vec{k}, \vec{D} \text{ and } \vec{E} \text{ are in the same plane} \]  

(10)

No we need to find the magnitude and possible orientations of the vector \( \vec{k} \) that would make the plane waves given in (4) compatible with the ME.

Replacing (9) in (8),

\[
\varepsilon_0 c^2 \vec{k} \times \left( \frac{1}{\omega} \vec{k} \times \vec{E} \right) = -\omega \vec{D}
\]

\[
\vec{k} \times (\vec{k} \times \vec{E}) = -\frac{\omega^2}{\varepsilon_0 c^2} \vec{D}
\]

\[
(\vec{k} \cdot \vec{E}) \vec{k} - (\vec{k} \cdot \vec{E}) \vec{k} = -\frac{\omega^2}{\varepsilon_0 c^2} \vec{D}
\]

\[
\frac{\vec{k} \cdot \vec{E}}{\vec{k} \cdot \vec{k}} \vec{k} - \vec{E} = -\frac{1}{k \cdot k \varepsilon_0 c^2} \vec{D}
\]

(11)

Notice, for a plane-wave \( \vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \) we expect it travels with a phase velocity

\[ v = \frac{\omega}{k} \]  

( where \( k^2 = \vec{k} \cdot \vec{k} \) ), to which we associate an index of refraction \( n = \frac{c}{v} = \frac{c}{\omega} k \).

Conveniently we write,

\[ n_k^2 = \frac{c^2}{\omega^2} \vec{k} \cdot \vec{k} \]  

(12)

Using this expression of \( n_k \), and in terms of the unit vector \( \hat{k} \) pointing in the direction of \( \vec{k} \), i.e.

\[ \vec{k} = \hat{k} \]  

(13)

, expression (11) takes the form,
\((\mathbf{k} \cdot \mathbf{E}) \mathbf{k} - \mathbf{E} = -\frac{1}{\varepsilon_0 n_k^2} \mathbf{D}\) \hspace{1cm} (14)

Here the unknowns are the orientation of the unit vector \(\mathbf{k}\) and the magnitude of \(n_k^2\).

In the most general case,
\[ D_j = \sum_i \varepsilon_{ij} E_i \quad j, i = 1,2,3 \]

However, there exists a system of reference (the principal axis reference) in which,
\[ D_j = \varepsilon_j E_j \quad \text{for} \quad j = 1,2,3 \hspace{1cm} (15) \]

with their corresponding associated indices of refraction,
\[ n_j^2 = \frac{\varepsilon_j}{\varepsilon_o} \hspace{1cm} (16) \]

Working in the principal axis reference, expression (14) defines a set of three equations

\[
(\mathbf{k} \cdot \mathbf{E}) \mathbf{k}_j - E_j = -\frac{\varepsilon_j}{\varepsilon_o n_k^2} E_j
\]

\[
(\mathbf{k} \cdot \mathbf{E}) \mathbf{k}_j - E_j = -\frac{n_j^2}{n_k^2} E_j \quad \text{for} \quad j = 1,2,3 \hspace{1cm} (17)
\]

More explicitly,

\[
(\mathbf{k}_j E_j + \mathbf{k}_l E_l + \mathbf{k}_m E_m) \mathbf{k}_j - E_j = -\frac{n_j^2}{n_k^2} E_j
\]

\[
\mathbf{k}_j^2 E_j + \mathbf{k}_j \mathbf{k}_l E_l + \mathbf{k}_j \mathbf{k}_m E_m - E_j = -\frac{n_j^2}{n_k^2} E_j
\]

\[
\left(\frac{n_j^2}{n_k^2} + \mathbf{k}_j^2 - 1\right) E_j + \mathbf{k}_j \mathbf{k}_l E_l + \mathbf{k}_j \mathbf{k}_m E_m = 0
\]

\[
\left(\frac{n_j^2}{n_k^2} - \mathbf{k}_l^2 - \mathbf{k}_m^2\right) E_j + \mathbf{k}_j \mathbf{k}_l E_l + \mathbf{k}_j \mathbf{k}_m E_m = 0
\]

More explicitly,

\[
(\frac{n_1^2}{n_k^2} - \mathbf{k}_2^2 - \mathbf{k}_3^2) \mathbf{E}_1 + \mathbf{k}_1 \mathbf{k}_2 \mathbf{E}_2 + \mathbf{k}_1 \mathbf{k}_3 \mathbf{E}_3 = 0
\]

\[
\mathbf{k}_2 \mathbf{k}_1 \mathbf{E}_1 + \left(\frac{n_2^2}{n_k^2} - \mathbf{k}_1^2 - \mathbf{k}_3^2\right) \mathbf{E}_2 + \mathbf{k}_2 \mathbf{k}_3 \mathbf{E}_3 = 0
\]
\[ \hat{k}_1 \hat{k}_1 E_1 + \hat{k}_1 \hat{k}_2 E_2 + \left( \frac{n_3^2}{n_k^2} - \hat{k}_1^2 - \hat{k}_2^2 \right) E_3 = 0 \]

There will be a solution for \( E_1, E_2, E_3 \), different from zero if the determinant of the ordered coefficient is equal to zero.

\[
\begin{vmatrix}
\left( \frac{n_1^2}{n_k^2} - \hat{k}_2^2 - \hat{k}_3^2 \right) & \hat{k}_1 \hat{k}_2 & \hat{k}_1 \hat{k}_3 \\
\hat{k}_2 \hat{k}_1 & \left( \frac{n_2^2}{n_k^2} - \hat{k}_1^2 - \hat{k}_3^2 \right) & \hat{k}_2 \hat{k}_3 \\
\hat{k}_3 \hat{k}_1 & \hat{k}_3 \hat{k}_2 & \left( \frac{n_3^2}{n_k^2} - \hat{k}_1^2 - \hat{k}_2^2 \right)
\end{vmatrix} = 0
\]

For a given direction of propagation (i.e. for a given set of \( \hat{k}_1, \hat{k}_2, \hat{k}_3 \)) this system of three equations give an equation for \( n_k^2 \).

Although it would appears that the resulting equation for \( n_k^2 \) is cubic, “it turns out it is quadratic; thus the result are two values for \( n_k \).