LATERAL WAVES: Conical acoustic waves at an interface

The figure shows an XY plane interface. A source at Q is located along the z-axis in the, for example, air-medium. At \( t=0 \) the source emits spherical waves advancing at speed \( c_1 \) (while travelling in the medium-1), and reaching a denser medium-2 (water or stainless steel) where the sound propagates faster at speed \( c_2 \). [For example, in air \( c_1=340 \text{ m/s} \); in water \( c_2=1443 \text{ m/s} \); in stainless steel \( c_2=5980 \text{ m/s} \).] Q is located at an arbitrary distance \( b \) from the interface (\( b \) need not be large compared with the wavelength \( \lambda \)).

The reflected spherical wave-front can be obtained by assuming a fictitious source located at \( Q' \) (the latter is a mirror image of Q.) The points P (on the reflected wave-front) and U (on the refracted wave-front) are reached simultaneously at the time \( t \). The rays QS, SP and SU satisfy the Snell’s law.
Drawing the **direct wave wave-fronts**

For a given time \( t \), we draw a spherical surface with radius \( c_1 t \) with center at \( Q \), as shown in Fig.1. This constitutes the wavefront of the direct wave at time \( t \).

**Drawing the ordinary reflected wave wave-fronts**

Let’s consider an arbitrary point \( P \) reached by the ray \( QSP \) in a time \( t \). (\( QS \) and \( QP \) satisfy the Snell’s law). The path \( QSP \) is travelled in a time \( t \) with speed \( c_1 \). A collection of points reached by reflected rays at the time \( t \) constitutes a wave-front of the ordinary reflected wave-front. Below we describe a simple geometrical procedure to draw such a reflected wave-front.

Let’s recall that when we applied the principle of least time to rays reflected from the interface (Section 11.2), we concluded that the trace of the “actual” reflected ray could be obtained as if it were originated from a fictitious source located at \( Q' \) (where \( Q' \) was located at the mirror image of \( Q \)). That is, in Fig.1 above, the ray \( Q'S \) can be considered being an image of the ray \( QS \) (i.e. both travel at the same speed \( c_1 \)). Therefore, the distance \( Q'P \) is equal to \( c_1 t \). Accordingly, the ordinary wave-front can be obtained by drawing a sphere of radius \( c_1 t \) with center at \( Q' \), as shown in Fig.2 below.

As an aid to the eye (and facilitate the drawings of the direct and reflected wave-fronts when starting from scratch), a point \( L \) has been drawn at the interface (see Fig.2 below). That point is reached simultaneously at the time \( t \) by a ray emitted from \( Q \) and by a ray emitted from the fictitious source \( Q' \). The direct wave wave-front can then be drawn as a sphere with radius \( QL \) with center at \( Q \); the ordinary reflected wave wave-front can be drawn as a sphere with radius \( Q'L \) with center at \( Q' \).

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**Fig.2** Direct wave wave-front drawn as a sphere with radius \( QL \) with center at \( Q \). Ordinary reflected wave wave-front drawn as a sphere with radius \( Q'L \) with center at \( Q' \).
**Drawing the refracted wave wave-fronts**

Arbitrarily we chose a value of \( b \) numerically equal to \( c_1 \).

First, consider first a ray that is incident normal to the interface, and choose a total travel time is equal to, for example, 2 seconds (1 s travelling in the water with speed \( c_1 \), 1 sec travelling in the solid with speed \( c_2 \).) This determines one point of the refracted wave-front.

Then, for any other incident ray like QM (hitting the surface at a given distance \( x_M \) from the origin,) determine the time of travel \( t_{QM} \). Since the total travel time is 2 seconds, the length of the ray MU is \( c_2(2 - t_{QM}) \). The orientation of MU (which depends on the orientation of QM, is determined by the Snell’s law. Repeating this process for different points M, the refracted wave-front can be obtained. See schematics in Fig.3 below. Detailed on-scale construction of the refracted wave-front is left as an assignment.

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**Fig. 3** Suggested diagram for the construction of the refracted wave-front.

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**Extraordinary reflected wave-fronts**

In contrast to the one-point reflection rays addressed in the sections above (that allowed constructing the ordinary reflected wave-front), this time consider alternative reflection paths, like those paths that partially propagate in medium-1 and partially propagate along the interface. More precisely, consider an arbitrary path that leaving Q, reaches the interface at X, propagates along the interface with speed \( c_2 \) until it reaches an (arbitrary) fixed point W, and then goes to P. This new alternative path is shown in Fig.4
To simplify the problem, let’s break it down into two partial analysis. For example, let’s consider first the following variational problem (see Fig.5). For arbitrary fixed points Q and S, what is the location of the point X that makes the time of travel an extreme?

Formally, what we have to do is to find the stationary values of the function,

$$ t(QXS) = \frac{QX}{c_1} + \frac{XS}{c_2} = t(\alpha) $$

which leads to the following result: the path QXS is an extreme when the angle of incidence $\alpha$ satisfies

$$ \sin (\alpha_{\text{extreme}}) = \frac{c_1}{c_2} $$
That is, QXS is an extreme when its angle of incidence is equal to the total internal reflection angle $\alpha_0$, which is given by $\sin(\alpha_0) = c_1/c_2$.

One can argue then that, for a given fixed point, the path that makes the optical path length stationary is $QM_1M_2P$ indicated in Fig.6. This ray reaches the point $P$ at a time $t'$ that is lower than $t$.

**Conical wave-fronts**

If point $P$ is reached by the ray $QM_1M_2P$ in a given time $t'$, let’s find out other points $P'$ in the medium-1 that are reached by a ray $QM_1M_2'P'$ at the same time $t'$.

It turns out, the locus of such points $P'$ lie along a line that a) passes through the given point $P$, and b) makes an angle $\alpha_0$ with the horizontal. (This can be demonstrated using the similarity properties of triangles, and it is left as an assignment.)

**Hint:** Notice, the difference between $t(QM_1M_2P)$ and $t(QM_1M_2'P')$ is given by, $\Delta t = (M_2P)/c_1 - (M_2M_2')/c_2 - (M_2'P')/c_1$. Using the similarity properties of triangles demonstrate that this difference is identically equal to zero.

Notice, these locus of point $P$, $P'$,..., constitute a wave-front of different shape than the ordinary spherical reflected wave-front (also drawn in Fig. 7 for comparison). These wave-fronts are known a conical waves.
Ordinary reflected wave-front (at time $t$)

Direct wave (at the time $t$)

Conical wave-front

Refracted wave

**Fig. 7** Illustration of the generation of conical waves.

1 http://www.kayelaby.npl.co.uk/general_physics/2_4/2_4_1.html