1. The figure shows a cross section of a long cylindrical conductor of radius \( a = 4 \text{ cm} \), which carries a total current of \( 4\pi \text{ amp} \). The current is flowing out of the page. The magnitude of the current density \( J(r) \) is given as a function of radial distance from the center of the wire’s cross section as
\[
J(r) = J_o \left( 1 - \frac{r}{a} \right) .
\]

1A (12 points)
Calculate the value of \( J_o \).
Make a plot of \( J(r) \) vs \( r \).

1B (12 points)
Provide an expression for the magnitude of the magnetic field \( B(r) \), for points where \( r < a \).
Calculate the value of \( r \) at which the magnetic field \( B \) is maximum.
Make a plot of \( B(r) \) vs \( r \).

1C (12 points)
Provide an expression for the magnitude of the magnetic field \( B(r) \), for points where \( r > a \).
Calculate the value of \( r \) at which the magnetic field \( B \) is maximum.
Make a plot of \( B(r) \) vs \( r \).
2. A square loop of wire PSUT with side $a = 2$ cm carries an current $I_2 = 3$ mA. The loop is placed near an infinitely long wire carrying current $I_1 = 10$ mA, as shown in the figure. The distance from the long wire to the center of the loop is also “a”.

![Diagram of a square loop and a long wire](image)

2A **(12 points)** Calculate the net vector force acting on the square loop.
(Express the force in term of the unit vectors $\hat{i}, \hat{j}, \hat{k}$).

2B **(12 points)** Calculate the vector force acting on the segment PS.
(Express $\vec{\mu}$ in term of the unit vectors $\hat{i}, \hat{j}, \hat{k}$).

2C **(6 points)** If the magnitude of the current $I_1$ starts to decrease, indicate the direction of the induced current in the square loop:

PSUT or TUSP?

Circle your answer.
3. The figure shows a rectangular (PSUT coil of wire, of dimensions 10 cm by 5 cm. It carries a current $i$ of 1 Ampere s hinged along one long side. It is mounted in the $xy$ plane, at $30^\circ$ to the direction of a uniform magnetic field of magnitude 1 Tesla. (Notice that the magnetic field exists all over the space.)

3A (12 points) Calculate the vector magnetic dipole $\vec{\mu}$ (Express $\vec{\mu}$ in term of the unit vectors $\hat{i}, \hat{j}, \hat{k}$).

Express also the magnetic field $\vec{B}$ in term of the unit vectors $\hat{i}, \hat{j}, \hat{k}$.

3B (12 points) Calculate the vector magnetic-torque acting on the loop.

3C (12 points) What is the magnetic-force vector acting on the segment SU whose length is $|SU| = 10$ cm).
Some formulas:

- \( \mu = 10^{-6} \) \( \text{nano} = 10^{-9} \)
- \( \ln(ab) = \ln(a) + \ln(b) \) \( \ln(a/b) = \ln(a) - \ln(b) \)
- Electron mass: 9.1 \( \times 10^{-31} \) Kg
  Proton mass = 1.67 \( \times 10^{-27} \) Kg
- Electron charge 1.6 \( \times 10^{-19} \) C
- 1 Gauss = 10^{-4} Tesla
- Centripetal acceleration: \( a_c = \frac{v^2}{R} \)
- \( 1 \text{eV} = 1.6 \times 10^{-19} \) J

\[ E = \frac{q}{4\pi \varepsilon_0 \left( z^2 + R^2 \right)^{3/2}} \] (Field along the axis passing through the center of the ring)

- Gauss' Law \( \Phi = \oint_S E \cdot d\vec{s} = q/\varepsilon_0 \), where \( q \) is the net charge inside the Gaussian surface \( S \)
- Definition of Electric Potential \( V(r) = \frac{W_{\text{ext}}(\infty q_0 \to r)}{q_0} \)

Assuming the potential at infinity is zero

**ELECTRICITY**

- Coulomb's Law: \( \vec{F} = \frac{1}{4\pi \varepsilon_0} \frac{q_2 q_1}{r^2} \hat{u}_r \)
- Electric filed for an infinite and uniformly-charged sheet: \( E = \sigma/2\varepsilon_0 \) \( \sigma \) is the surface charge density.

For a uniformly charged ring with radius \( R \) and total charge \( q \):

\[ E(z) = \frac{1}{4\pi \varepsilon_0} \frac{q z}{(z^2 + R^2)^{3/2}} \] (Field along the axis passing through the center of the ring)

- Electric charge 1.6 \( \times 10^{-19} \) C
- Definition of Electric Potential difference: 
  \[ V(A) - V(B) = \frac{W_{\text{ext}}(B \to q \to A)}{q_0} \]

- Electric potential due to a point charge \( q \): 
  \[ V(r) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r} \] (Assuming the potential at infinite is zero)

- Potential difference: 
  \[ V_f - V_i = -\int_i^f E \cdot d\vec{r} \]

- Relationship between \( E \) and \( V \): 
  \[ E_x = -\frac{dV}{dx} \]

- About capacitance
  \[ Q = CV \quad U = CV^2 / 2 = Q^2 / 2 \frac{C}{C} \]

  For a capacitor of parallel plates: 
  \[ C = \frac{A \varepsilon_0}{d} \]

- RC circuit: Time constant \( \tau = RC \)

**MAGNETISM**

- \( \mathbf{F} = q \mathbf{v} \times \mathbf{B} \)  \( \mathbf{F} = \) force, \( q = \) charge, \( \mathbf{v} = \) velocity, \( \mathbf{B} = \) magnetic field

- Magnetic field produced by a charge \( q \) that moves with velocity \( \mathbf{v} \)
  \[ \mathbf{B} = \frac{\mu_0}{4\pi} \frac{q}{r^3} \mathbf{v} \times \mathbf{r} \]

- \( \Phi = L i \)  \( \Phi = \) Magnetic flux, \( L = \) inductance, \( i = \) current

- Hall effect \( BI = nqiV_{\text{Hall}} \)

- Inductive reactance \( X_L = \omega L \)

- \( B = \frac{\mu_0 I}{4R} \) Magnetic field at the center of a semi-circle of radius "R"

- \( B = \frac{\mu_0 I\phi}{4\pi R} \) Magnetic field at the center of an arc of angle \( \phi \) (in radians) and
radius "R".

- \( B = \frac{\mu_0 I}{2\pi r} \) \quad Magnetic field produced by a infinitely long wire at a distance "r" from it.

- \( \frac{F}{L} = \mu_0 \frac{I_a I_b}{2\pi d} \) \quad Force per unit length between two parallel long wires, carrying currents \( I_a \) and \( I_b \) respectively, separated by a distance "d"

- Ampere’s Law \( \int_B \cdot ds = \mu_0 I \) \quad Here \( I \) in the current encircled by \( C \).

- Faraday's Law \( \varepsilon = -\frac{\partial \Phi}{\partial t} \), where \( \Phi \) = Magnetic flux and \( \varepsilon \) = electromotive force

- Definition of the magnetic dipole moment of a loop of area \( A \), carrying a current \( I \): \( \mu = I A \hat{n} \) where \( A \) = area, \( I \) = current, \( \hat{n} \) = unit vector perpendicular to the loop
\[ i = \int j'(r) \, dr \]
\[ = \int_{0}^{a} j'(r) \, 2\pi r \, dr \]
\[ = \int_{0}^{a} j_0 (1 - \frac{r}{a}) \, 2\pi r \, dr \]
\[ = 2\pi j_0 \left\{ \int_{0}^{a} r \, dr - \int_{0}^{\frac{a}{2}} r \, dr \right\} \]
\[ \left\{ \frac{a^2}{2} - \frac{1}{3} a^3 \right\} \]
\[ \left\{ \frac{1}{6} a^2 \right\} \]

\[ i = 2\pi j_0 \frac{1}{6} a^2 = \frac{1}{3} \pi j_0 a^2 = i' \]

\[ i = 4\pi \]
\[ a = 4 \times 10^{-2} \]
\[ \Rightarrow j_0 = \frac{3 i'}{\pi a^2} = \frac{3 \times 4\pi}{\pi \times (4 \times 10^{-2})^2} = \frac{\frac{3}{4}}{16 \times 10^{-4}} \]

\[ j_0 = \frac{3}{4} \times 10^4 \frac{A m}{m^2} \]

Plot of \( j_0 (1 - \frac{r}{a}) \) vs \( r \)
1B) For \( r < a \)

\[ 2\pi r \quad B(r) = \mu_0 \quad i_{\text{inside}} \]  \( \text{(1)} \)

\[ i_{\text{inside}} = \int j'(s) \, ds \]

\[ = \int_0^r j'(s) \, 2\pi s \, ds \]

\[ = \int_0^r j_o \, (1 - \frac{s}{a}) \, 2\pi s \, ds \]

\[ = 2\pi j_o \int_0^r (1 - \frac{s}{a}) s \, ds \]

\[ \left\{ \int_0^r s \, ds - \int_0^r \frac{s^2}{a} \, ds \right\} \]

\[ \frac{r^2}{2} - \frac{1}{3a} \gamma^3 \]  \( \text{(2)} \)

\[ 2\pi \gamma B(r) = \mu_0 \quad \gamma \left( \frac{1}{2} r - \frac{1}{3a} r^2 \right) \quad \pi j_o \]

\[ B(r) = \mu_0 j_o \left( \frac{1}{2} r - \frac{1}{3a} r^2 \right) \]
\( B \) is max when \( \frac{dB}{dr} = 0 \)

\[ \frac{dB}{dr} = \mu_0 J_0 \left( \frac{1}{2} - \frac{2}{3a} r \right) \]

\[ \frac{dB}{dr} = 0 \Rightarrow \frac{1}{2} = \frac{2}{3a} R \]

\[ R = \frac{3}{4} a \]

\( B \) is max at \( R = \frac{3}{4} a \)

1c \) For \( r > R \)

\[ B(r) = \frac{\mu_0 I}{2\pi r} \]

\( B \) is max at \( r = a \).
2A) \[ \vec{F} = I \vec{L} \times \vec{B} \]

- On segment TP:
  \[ \vec{F}_{TP} = I_2 \, \vec{TP} \times \vec{B}_s \]
  Points in the \( \hat{j} \) direction

  \[ F_{TP} = I_2 \, a \, \frac{M_0 \, I_1}{2 \pi \, \text{distance}} \]

  \[ = I_2 \, a \, \frac{M_0 \, I_1}{2 \pi \, \frac{a}{2}} = \frac{1}{\pi} M_0 I_1 I_2 \quad (2) \]

- On segment SU
  \[ \vec{F}_{SU} = I_2 \, \vec{SU} \times \vec{B}_s \]
  Points in the \( \hat{j} \) direction

  \[ F_{SU} = f_2 (a) \left( \frac{M_0 \, I_1}{2 \pi \, \text{distance}} \right) \]

  \[ = I_2 \, a \, \frac{M_0 \, I_1}{2 \pi \, \frac{3}{2} a} = M_0 \, \frac{I_1 I_2}{3 \pi} \quad (4) \]

- From (2) and (4)
  \[ \vec{F}_{TP} = - \frac{1}{\pi} M_0 \, I_1 I_2 \, \hat{j} \]
  \[ \vec{F}_{SU} = \frac{1}{3 \pi} M_0 \, I_1 I_2 \, \hat{j} \]
  \[ \vec{F}_{TP} + \vec{F}_{SU} = - \frac{2}{3} \frac{M_0 \, I_1 I_2}{\pi} \, \hat{j} \quad (5) \]
\[ \vec{F}_{T} + \vec{F}_{J} = - \frac{2}{3} \frac{\mu_{0}}{4\pi} 4 I_1 I_2 \hat{j} \]
\[ = - \frac{2}{3} 10^2 4 \times (3 \times 10^{-3}) (10 \times 10^{-3}) \hat{j} \]
\[ = - 8 \times 10^{-2} N \hat{j} \quad (6) \]

The magnetic field at the point of coordinate \( y \) is

\[ B(y) = \frac{\mu_{0} I_1}{2\pi y} \]

\[ dF = I_2 (dy) B(y) \]
\[ = I_2 \frac{\mu_{0} I_1}{2\pi y} dy \]

\[ F_{PS} = \int_{\frac{a}{2}}^{\frac{3}{2}a} dF = \int_{\frac{a}{2}}^{\frac{3}{2}a} I_2 \frac{\mu_{0} I_1}{2\pi} \frac{dy}{y} = \frac{\mu_{0} I_1 I_2}{2\pi} \int_{\frac{a}{2}}^{\frac{3}{2}a} \frac{dy}{y} \]
\[ = \frac{\mu_{0} I_1 I_2}{2\pi} \ln \frac{3a}{a} - \ln \frac{\frac{a}{2}}{\frac{a}{2}} \]

\[ F_{PS} = \frac{\mu_{0} I_1 I_2}{2\pi} \ln 3 \]
\[
F_{ps} = 2 \frac{M_0}{g \pi} I_1 I_2 \text{ m}^3 \\
= 2 \times 10^{-2} \times 3 \times 10^{-3} \times 10 \times 10^{-3} \text{ m}^3 \\
= 6 \times 10^{-12} \text{ m}^3 = 6.52 \times 10^{-12} \text{ Newtons} \\
\mathbf{F}_{ps} = (-6.6 \times 10^{-12} \text{ Newtons}, 0, 0) \\
\text{or} \\
\mathbf{F}_{ps} = -6.6 \times 10^{-12} \text{ N} \hat{z}
\]

2c)

\[ \text{Flux decreasing} \]

\[ \text{I}_1, \text{ decreasing} \]

Therefore, induced should respond reinforcing the field

\[ \text{induced is in the PSUT direction} \]
3A) \[ \vec{\mu} = \iota A \times R \]
\[ = (1\text{Am}) \times (5 \times 10^{-2} \times 10 \times 10^{-2}) \times \hat{R} \]
\[ \vec{\mu} = -5 \times 10^{-3} \text{Am} \cdot \text{m}^2 \hat{R} \]
\[ \vec{\mu} = (0, 0, -5 \times 10^{-3} \text{Am} \cdot \text{m}^2) \]

3B) \[ \vec{B} = IT \cos 30^\circ \hat{x} + IT \sin 30^\circ \hat{R} \]

3B) \[ \vec{\tau} = \vec{\mu} \times \vec{B} \]
\[ = (-5 \times 10^{-3} \hat{R}) \times (\cos 30^\circ \hat{x} + \sin 30^\circ \hat{R}) \]
\[ = -5 \times 10^{-3} \cos 30^\circ \hat{x} \times \hat{R} \]
\[ = -5 \times 10^{-3} \cos 30^\circ \hat{x} \]
\[ \vec{\tau} = -4.33 \times 10^{-3} \text{N} \cdot \text{m} \hat{z} \]
\[ \text{or} \]
\[ \vec{\tau} = (0, -4.33 \times 10^{-3} \text{N} \cdot \text{m}, 0) \]

3C) \[ \vec{F}_{\text{su}} = \iota S \vec{u} \times \vec{B} \]
\[ = (1\text{Am}) \times (-10 \times 10^{-2} \hat{j}) \times (\vec{B}) \]
\[ = -0.1 \hat{j} \times (\cos 30^\circ \hat{x} + \sin 30^\circ \hat{R}) \]
\[ = -0.1 (\cos 30^\circ \hat{x} \times \hat{R} + \sin 30^\circ \hat{R} \times \hat{R}) \]
\[ = -0.1 \sin 30^\circ \hat{x} + 0.1 \cos 30^\circ \hat{R} \]
\[ \vec{F}_{\text{su}} = -0.05 \hat{x} + 0.086 \hat{R} \]
\[ \text{or} \]
\[ \vec{F}_{\text{su}} = (-0.05 \text{N}, 0, 0.0086) \]