sum/difference sideband emerge in such an optomechanical system due to the nonlinear terms, and the sum/difference sideband generation can be greatly enhanced via achieving the matching conditions (which is quite similar to sum/difference frequency generation in a nonlinear medium). These perturbative signals of optomechanical nonlinearity should be important in understanding the nonlinear optomechanical interactions, especially in the parameter configuration of optomechanically induced transparency that interference phenomenon takes place. On the application side, these sideband signals may provide measurement of electric charge (or other weak forces) with higher precision (see Sec. V B).

It is well known in strong field nonlinear optics that there is high-order harmonic generation in intense driven atomic system with typical non-perturbative spectral structure: the spectrum decreases rapidly for the first few order sidebands, followed by a plateau where all the harmonics have the same strength, and ends up with a sharp cutoff. An analogon to atomic high-order harmonic generation is found in optomechanical system based on the optomechanical nonlinearity. In the parameter configuration of optomechanically induced transparency, if the pump power is strong enough, spectral structure output from the optomechanical system shows a similar non-perturbative structure: the spectral linearity, if the pump power is strong enough, spectral structure output from the optomechanical system shows a similar non-perturbative structure: the spectrum decreases rapidly for the first few order sidebands, followed by a plateau where all the sidebands have the same strength, and ends up with a sharp cutoff [see Fig. 6(a)]. Besides the same typical spectral structure, both high-order sideband generation and high-order harmonic generation also exhibit carrier-envelope phase-dependent effects in the few-cycle regime [see Figs. 6(a)–6(d) for different carrier envelope phases]. To reduce the pump power for observing high-order sideband generation, optomechanical dynamics in a wavelength-sized GaAs disk resonator is discussed. The linewidth of the sideband spectra can be estimated by the time-frequency uncertainty relation ΔωΔt ~ 2π.

In consideration of the quantum nature of the optomechanical interaction, many important phenomena arising from the quantum nonlinearity have been revealed, such as quantum nonlinear optomechanically induced transparency, which exhibits distinct dependence on the dimensionless parameter (it is also called “quantum parameter”) g₀/κ from the standard (classical) optomechanically induced transparency.

The dimensionless parameter g₀/κ is called “quantum parameter” because g₀/κ ~ h⁻¹/² plays the same role of Planck’s constant, and quantum effects vanish if g₀/κ → 0. The standard optomechanically induced transparency is classical because it is determined by the combined parameter g₀|a|, which retains when g₀ → 0 while |a| is high enough. To observe quantum nonlinear optomechanically induced transparency, the intracavity fields should be weak (|a| ≪ 1), and the window width can be estimated as ~Γ/κ in this case (the yellow regions shown in Fig. 7). For quantum optomechanics, the thermal noise of mechanical motion becomes important and the dissipative dynamics can be well described by the master equation

$$\dot{\rho} = \frac{i}{\hbar}[\rho, H] + \mathcal{G}\rho,$$

where $\mathcal{G} = \kappa \{D[\hat{a}] + \Gamma_m(n_{th} + 1)D[\hat{b}] + \Gamma_m n_{th} D[\hat{b}^\dagger]\}$ with $D[\hat{a}] = i\hbar \hat{a}, D[\hat{b}] = \frac{1}{2}\hat{b}, \frac{1}{2} \hat{b}^\dagger - \frac{1}{2} \hat{b}^\dagger \hat{b}^\dagger \hat{b}$ being the Lindblad superoperator, $\hat{p}$ denotes the density matrix for the optical and mechanical mode, and $n_{th} = 1/\exp(h\omega_m/k_B T) - 1$ is the thermal occupancy of the mechanical bath at the environmental temperature T. In general, the master equation is nonlinear and can only be solved numerically in the time-domain.

Calculation results of optomechanically induced transparency for different g₀/κ (g₀|a| and κ are kept fixed) are shown in Fig. 7 which display quantum to classical crossover. At the first mechanical sideband $Ω ≈ Ω_m$ [Fig. 7(a)], the “OMIT signal” (defined as the difference of the normalized probe transmission $|\delta\tilde{a}|^2 = κ^2 (|\tilde{a}|^2 + |\tilde{A}_m|^2)/4$ with and without control laser) converges to the classical results for small quantum parameters g₀/κ. The “OMIT signal”...
becomes more pronounced as the quantum parameter $g_0/\kappa$ increases; however, the enhancement of the signal is not obvious due to the weakness of the quantum optomechanical nonlinearity. To give a clear signature of the quantum nonlinearity, Fig. 7(b) shows the "OMIT signal" at the second mechanical sideband which vanishes in the classical limit $g_0/\kappa \rightarrow 0$. The signal of quantum nonlinear optomechanically induced transparency is found to be visible even for coupling strengths $g_0 \approx 0.1 \kappa$. Up to now, most realizations of optomechanical interaction are still in the single photon weak coupling limit, thus the signal of quantum nonlinear optomechanically induced transparency would verify the quantum nonlinearity of the optomechanical interaction.\textsuperscript{50} From an application perspective, it has been shown that the optomechanically induced transparency at the second mechanical sideband could be a sensitive tool of measuring the average phonon number of the mechanical oscillator.

E. Double optomechanically induced transparency

Electromagnetically induced transparency is of crucial importance in controlling the propagation and absorption of light due to the dramatic enhancement of optical nonlinearity. However, electromagnetically induced transparency in three-level atomic systems allows only a single transparency window in the frequency domain, which limits the manifold functionalities for nonlinear optics and optical communication.\textsuperscript{52} To resolve this problem, related configurations and effects have been studied in various multiple-level systems,\textsuperscript{53,54} and double electromagnetically induced transparency phenomenon arising from two-photon absorption has been proposed in a N-type four-level system composed of cold Rb atoms.\textsuperscript{55,56} which extends conventional electromagnetically induced transparency with single transparency window to the one with double and even multiple transparency windows. Double electromagnetically induced transparency offers some novel physical insight and potential applications. It has been shown that double electromagnetically induced transparency can be used to realize double-channel optical communication,\textsuperscript{56} high-resolution spectroscopy,\textsuperscript{57} tunable cross-phase modulation,\textsuperscript{58} and high-speed optical switches.\textsuperscript{59}

Similar to the electromagnetically induced transparency, optomechanically induced transparency also faces the frequency limitation of the transparency window, and the study of the conventional optomechanically induced transparency is also extended to the multi-channel cases, that is, the double and multiple optomechanically induced transparency.\textsuperscript{60} Compared to the conventional optomechanically induced transparency, double and multiple optomechanically induced transparency own many favorable features except multi-channel transparency windows, including extensive integration and performance. Reference 61 has shown that double optomechanically induced transparency can be achieved in a simple system with two mechanical resonators. In this case, there are two resonant matching between the optical beat frequency and the mechanical motion in frequency space, which leads to two interference windows in optomechanical induced transparency. Such double optomechanical induced transparency can be generalized to the multi-channel one by introducing additional mechanical oscillators.\textsuperscript{62}

As an another important approach, double optomechanically induced transparency can be achieved by coupling the mechanical resonator to a two-level system,\textsuperscript{63} such as two-level atoms, superconducting qubit circuits, and two-level defects. The underlying mechanism is quite similar to the electromagnetically induced transparency in multiple-level systems. The conventional optomechanically induced transparency of a single transparency window can be well explained by a typical A-type three-level system which is composed of three states $|0_a, 0_b\rangle$, $|0_a, 1_b\rangle$, and $|1_a, 0_b\rangle$ (as shown in Fig. 8), with the subscripts a and b represent the photonic and phononic states, respectively. The energy level of the mechanical resonator $|0_a, 1_b\rangle$ is split into two dressed states $|0_a, 1_b^+\rangle$ and $|0_a, 1_b^-\rangle$ when the mechanical resonator is resonantly coupled to the two-level system, and the effective resonant frequency of the mechanical resonator becomes $\omega_p - g$ and $\omega_p + g$ with $g$ being the coupling strength between the mechanical resonator and the two-level system. In this case, the optomechanical system is transformed into a four-level system and two transparent windows arise due to the resonant matching between the optical beat frequency and the effective mechanical frequency. It is also shown in Ref. 64 that the third-order nonlinear absorption is enhanced by constructive quantum pathway interference, while the linear absorption is inhibited by destructive quantum pathway interference. Due to the adjustability of the two-level system, the double optomechanically induced transparency becomes tunable and consequently paved an avenue towards the relevant applications.

Tunable double optomechanically induced transparency has also been studied in a hybrid optomechanical system with Coulomb coupling.\textsuperscript{65} The hybrid optomechanical system is composed of an optical cavity and two charged nanomechanical resonators, where cavity fields interact with one nanomechanical resonator via radiation pressure, while the two nanomechanical resonators are coupled with Coulomb force. Because of the Coulomb coupling, there are two dressed states for the mechanical resonators and the A-type three-level structure is split into double-A configuration when the resonance condition is met.

![FIG. 8. Level scheme of double optomechanically induced transparency](image-url)