CHAPTER 3

The ORIGINS of QUANTUM PHYSICS

3.1  BLACKBODY RADIATION

3.1.A  The Kirchoff’s Law and the concept of blackbody radiator

3.1.A.a  Macro-scale heat-transfer concepts
  Heat transfer, radiation, absorbed radiation, blackbody radiator.

3.1.A.b  Kirchhoff’s law
  Connection between the emissivity and absorptivity of radiation by a body.

3.1.A.c  Cavity’s aperture as an approximation to a blackbody radiator.

3.1.B  Radiation in a cavity

3.1.B.a  Emission of radiation by an accelerated charge
  Emission of radiation by an oscillating charge

3.1.B.b  Light Scattering
  Absorption and re-emission of energy
  Fraction of the incident energy scattered by the charge
  Scattering cross section

3.1.B.c  Radiation damping
  Rate at which the oscillator looses energy

3.1.C  Radiation and thermal equilibrium

3.1.C.a  Light intensity spectral density $I(\omega)$ at equilibrium

3.1.C.b  Classical calculation of the atom’s average energy $W$
  The Boltzmann’s distribution
  The equipartition theorem
  The ultraviolet catastrophe

3.1.C.c  Light energy density $U(\omega)$ at equilibrium. The concept of electromagnetic modes

3.1.C.d  The ultraviolet catastrophe

3.1.D  The birthday of Quantum Physics: Planck’s Hypothesis to calculate the atom’s average energy $W$
  Quantum physics view to calculate average energy. Oscillator’s energy quantization.

3.2  PARTICLE-LIKE PROPERTIES of RADIATION

3.2.A  Processes involving the absorption or scattering of radiation by particles
3.2.Aa Photoelectric effect. Einstein’s hypothesis of the quantized radiation energy
3.2.Ab Compton effect. Scattering of a complete quantum of radiation in definite direction.
3.2.Ac Pair production
3.2.B Processes where interacting charged particles produce radiation
   3.2.Ba Bremsstrahlung
   3.2.Bb Pair annihilation

3.3 WAVE-LIKE PROPERTIES of PARTICLES
   3.3.A The de Broglie Hypothesis
   3.3.B Experimental confirmation of the de Broglie hypothesis. Electron Diffraction

3.4 WAVE-PARTICLE DUALITY

References
Richard Feynman, “The Feynman Lectures on Physics,” Volume I, Chapter 32 (Sections 32-1 to 32-3), Chapter 41 (Section 41-2, 41-3), and Chapter 24 (Section 24-1 and 24-2).
CHAPTER 3
THE ORIGINS OF QUANTUM PHYSICS

- By the end of the 19th century, classical physics appeared to be on solid foundations. The universe was conceived as containing matter consisting of particles, obeying Newton’s law of motion; Einstein’s theory of special relativity generalized classical physics to include the region of high velocities.

and

radiation (waves) obeying Maxwell’s equation of electromagnetism.

- Some key experiments, however, could not be explained by classical physics,
  Blackbody Radiation
  Photoelectric Effect
  Compton Effect
  which forced the introduction of radical new concepts:
  Quantization of physical quantities
  Particle properties of radiation
  Wave properties of particles

In the previous chapter we reviewed the strength of classical physics (electromagnetism and relativity.) In this chapter we aim to familiarize with the topics that forced a transition from classical to quantum concepts.

3.1 BLACKBODY RADIATION

On December 14, 1900, at a meeting of the German Physical Society, Max Planck presented his paper “On the Theory of the Energy Distribution Law of the Normal Spectrum.” This date is considered to be the birthday of quantum physics. He broke ranks with the then quite accepted “law of equipartition energy” that stated that the average energy of, for example, a harmonic oscillator was independent of its natural frequency. He instead postulated a frequency-dependent average energy and, very revolutionary, that the energy could have only discrete (quantized) values. Planck presented his paper while trying to explain the observed properties of thermal radiation. In this section we will review that development, as to gain some grasp of why the discreetness of a physical property lead to a radical different behavior of matter.

The problem that Plank had at hand was related to thermal radiation. Thermal radiation refers to the emission of electromagnetic waves by any object at a given temperature absolute temperature $T$. Matter in a condensed state (solid or liquid) emits a continuous spectrum of radiation. But even at temperatures as high as a few thousand of degrees Kelvin, most of the
radiation is invisible to us as they lie in the infrared part of the electromagnetic spectrum.

\[
\begin{array}{c|c|c|c}
\text{Radiation energy} & \text{Frequency } f (10^{14} \text{ Hz}) & \text{Temperature}^0 \text{K} \\
1000 & 2 & \\
1500 & 2 & \\
2000 & 2 & \\
\end{array}
\]

Fig 3.1 Spectral thermal radiance of a body showing the strong dependence on the body's temperatures.

With increasing temperature the body emits more radiation while the frequency of the most intense radiation (see dashed lines) also increases. The relationship between the temperature of a body and the frequency spectrum allows estimating the temperature of a hot body, such as a star, by observing the radiation it emits.

The detailed form of the spectrum depends somewhat upon the composition of the body. HOWEVER, there exist a certain type of body configuration that emit thermal spectra of a universal character. These are called blackbodies (they efficiently absorb all radiation incident upon them.) Independent of the details of their composition, it is found that

all blackbodies at the same temperature emit thermal radiation with the same spectrum

(1)

It turns out that,

- while the invariability of the spectrum from blackbody to blackbody can be explained based on thermodynamic arguments (see Kirchoff's law below, which states that the ratio of the emissive power \( P_e \) to the absorption coefficient \( a \) is the same for all bodies at the same temperature.)
- the specific form of the spectrum (tending to zero at very low and very high frequencies) can not be explained in terms of classical physics arguments. (Planck was able to solve this problem on the basis of quantum arguments.)

Section 3.1 describes in more detail these two aspects.
3.1.A The Kirchoff’ Law and the Concept of Blackbody Radiator

3.1.A.a Macro-scale heat-transfer concepts

Heat transfer

While heat transfer \( Q \) by conduction and convection requires a material medium, radiation does not.

Thermal Radiation

Refers to the emission of electromagnetic waves by any object at a given temperature absolute temperature \( T \).

Experimentally it was found (in 1879) that the rate at which a system emits radiation is proportional to the fourth power of its absolute temperature

\[
\frac{\Delta Q}{\Delta t} \sim T_{\text{object}}^4
\]

Since the rate of radiative heat flow is also proportional to the area of the object, the emitted power per unit area is typically expressed as,

\[
R_{\text{emitted}} = \frac{1}{A} \times \frac{\Delta Q}{\Delta t} = e \times \sigma \times T_{\text{object}}^4
\]

\( \sigma = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4 \), \( e \) is the Emissivity of the surface \((0 < e < 1)\)

Absorbed Radiation

A system absorbs radiation proportional to the fourth power of the absolute temperature of its surroundings.

\[
R_{\text{absorbed}} = a \times \sigma \times T_{\text{surrounding}}^4
\]

\( a \) indicates the relative ability of the surface to absorb radiation \((0 < a < 1)\)

For an object in equilibrium with its surrounding:

\[
R_{\text{emitted}} = R_{\text{absorbed}}, \text{ which implies } e = a
\]
Heat flow via radiation from an object of area A to its surroundings

\[
\frac{\Delta Q}{\Delta t} = e \ A \sigma \left( T_{\text{object}}^4 - T_{\text{surroundings}}^4 \right)
\]  

(5)

3.1.A.b Kirchhoff’s law

The emissive power \( P_e \) of a given material is defined as follows (see also Fig. 3.2A for identifying the variables being used)

\[
P_e(\vec{k}) \ d\omega \ d\Omega = \text{emitted power per unit area of the body, composed of electromagnetic radiation of frequency between } \omega \text{ and } \omega + d\omega, \text{ and directed into a solid angle } d\Omega \text{ about the direction } \vec{k}
\]

\[
k = |\vec{k}| = \omega / c
\]

where \( c \) is the speed of light.

\[
k = 2\pi / \lambda
\]

Fig. 3.2 Radiation of wave-vector around \( \vec{k} \) emitted by a surface at temperature \( T \).

G. R. Kirchhoff proved in 1859 by using general thermodynamic arguments that, at any given electromagnetic wavelength,

\[
\text{the ratio of the emissive power } P_e \text{ to the absorption coefficient } a
\]

is the same for all bodies at the same temperature

\[
\left. \frac{P_e}{a} \right|_1 = \left. \frac{P_e}{a} \right|_2 = \left. \frac{P_e}{a} \right|_3 = etc
\]

(6)

**Blackbody Radiator**

A perfect radiator would have an absorption coefficient \( a=1 \)

Such a system is called a blackbody radiator

(7)
Note: The “blackbody” name of originates from the fact that “black looking” objects are in fact objects that do not reflect any of the incident light. "Black color" is in fact an absence of color. BUT in the context of the quantum physics that we are developing here, a blackbody radiator may not appear black at all. In fact, a blackbody emits very efficiently all colors.

3.1.A.c Cavity’s aperture as an approximation to a blackbody radiator
A blackbody radiator is an idealization; however, a small orifice in a cavity closely approximates one.

![Fig. 3.3 Blackbody radiator.](image)

- Any radiation incident from the outside and passing through the hole will almost completely be absorbed in multiple reflections inside the cavity; that is, the hole has an effective absorption coefficient close to unity \(a=1\).
- Since the cavity is in thermal equilibrium, (the radiation getting inside through the small opening and that escaping from the small opening will be equal, That is, emissivity \(e=1\)),
  
  Then, we infer that the spectral characteristics of the radiation emitted by the aperture should have then the characteristics of a blackbody radiator.

3.1.B Radiation inside a cavity
Our next objective is to characterize the bath of electromagnetic radiation existent inside a blackbody radiator cavity. We would like to know,

- How much light energy of angular frequency \(\omega\) (across the electromagnetic spectrum) does exist inside the cavity at equilibrium conditions at a given temperature \(T\)?
Two approaches for answering this question, each arriving to the same general result, are worth mentioning (in this chapter we will address in very much detail only approach-1):

**APPROACH-1:** Feynman’s tackles the problem by considering an atom, of characteristic resonance frequency $\omega_o$, located inside the cavity and in equilibrium with the bath of electromagnetic radiation. The equilibrium refers to the fact that the atom absorbs energy from the electromagnetic bath and then releases it to the cavity. At equilibrium these two quantities must be equal.

![Fig. 3.5 Atom](image)

To this effect the atom is modeled as an electrical harmonic oscillator in equilibrium with an electromagnetic field. Accordingly, the equation of motion for the charge $q_e$ is given by,

$$m_e \frac{d^2 x}{dt^2} + m_e \gamma \frac{dx}{dt} + kx = q_e E_o e^{j\omega t}$$

(8)

Here the second term on the left side of the equation,

$$m_e \gamma \frac{dx}{dt},$$

accounts for the presence of a dissipation energy source, which, in our case, can be identified with the loss of energy $<\mathcal{P}>$ due to the electromagnetic radiation by the accelerated charge.

In the third term, the value of $k$ is chosen according to $k = m_e \omega_o^2$. 
The equilibrium inside the cavity is obtained by a balance between:

- **The rate at which the atom loses energy**, (using (8), the loss is expressed as:
  \[
  \frac{dW}{dt} = -\gamma W,
  \]
  where \( W \) is the average energy of the atom at a given time);

- **The rate of energy absorbed by the atom from the light energy existent inside the cavity (and re-emitted back to the cavity).**

Quantitatively, that rate is given by:

\[
\langle P \rangle = I(\omega_o) \frac{(2\pi)^2}{3\gamma} \left( \frac{q_e^2}{4\pi\varepsilon_0 m_e c^2} \right)^2 \omega_o^2,
\]

(as will be shown in the sections below.)

At an equilibrium condition, \( W \) should remain constant.

Let’s review in more detail this statement of energy balance.

If an oscillator is initially excited (shaking up and down) and then left alone with not additional excitation, the oscillator’s energy \( W \) would decrease due to the emission of electromagnetic radiation. When this situation is modeled through Eq (8), then one can account for that change in energy through the expression \( \frac{dW}{dt} = -\gamma W \). That is, the oscillator’s energy would decay exponentially with time.

![Fig. 3.6 Atom modeled as an electrical harmonic oscillator. If the atom were initially excited (optically or mechanically) and there were not additional excitation, the atom’s energy \( W \) would decrease due to the emission of electromagnetic radiation. That change in energy would occur at a rate given by \( \frac{dW}{dt} = -\gamma W \). That is it will decay exponentially with time.](image)
One way to avoid the atom loses its average energy $W$ is to enclose it in a box made of perfectly reflecting walls, as shown in Fig. 3.7 below. There would be, then, electromagnetic energy trapped in the cavity, which can supply back energy to the atom. That way the atom may absorb energy from the cavity, while still releasing energy at the rate $\gamma W$. But, how to calculate the energy absorbed by the atom from the cavity?

To calculate rate of energy absorbed by the atom from the incident radiation, we will appeal to the concept of **scattering**, a process in which the incident radiation drives the charge, and the latter re-emit it in all directions. This happens because accelerated charges emit radiation. Calculating the **emitted power** $<P>$ emitted by a charged-particle will allow, then, to calculate the **power absorbed by the atom from the incident radiation**; *they are the same*. The latter justifies the diagram in Fig. 3.7. [We will find out in the following sections that $<P> = I(\omega_o) \frac{2\pi^2}{3\gamma} \frac{q_e^2}{4\pi\varepsilon_0 m_e c^2} \omega_o^2$]

![Fig. 3.7](image-url) Atom (modeled as an electrical harmonic oscillator) emits electromagnetic radiation upon being excited by the electromagnetic field existent in the cavity. $I(\omega)$ stands for the intensity of the electromagnetic radiation of frequency $\omega$ existent in the cavity. Because the atom has a resonance frequency $\omega_o$, its absorption is more efficient at that frequency. Notice the greater $I(\omega_o)$ the greater the amount of energy absorbed by the atom and subsequently scattered.

In equilibrium, power absorbed $<P>$ from the radiation in the cavity should be equal to the rate at which the atom loses energy $\gamma W$

At equilibrium $<P> = \gamma W$ (9)
Since $< P > = I(\omega_o) \frac{(2\pi)^2}{3y} \left(\frac{q_e^2}{4\pi\epsilon_0 m c^2}\right)^2 \omega_o^2$ (as will be demonstrated in the sections below) the greater amount of light $I(\omega)$ at $\omega=\omega_o$, the greater the $< P >$. Hence,

If, for example, under different arguments we arrived to the conclusion that $W$ should have a constant value (for instance, $W = 1/2 kT$, as an elementary classical argument would suggest), that will force the value of $I(\omega_o)$ in (9) to adopt a particular specific value in order to satisfy such an expression; That is, $I(\omega_o)$ cannot be too high, it cannot be too low; the cavity has to contain radiation of frequency $\omega_o$ in an amount such that $<P>$ matches the value of $\gamma W$.

In short, expression (9) constitutes an analytical expression for the required Light Intensity Spectral Density $I(\omega)$ that must exist inside the cavity for an equilibrium situation to prevail. This is the answer to the problem posted in Fig. 3.4.

**APPROACH-2: (We will not address in detail this second approach in this chapter)** The second approach is a bit more abstract. It involves counting electromagnetic modes, and considers the energy-interchange equilibrium reached between the electromagnetic-modes inside the cavity and the cavity’s walls. This will be addressed in more detail in Section 3.5 “Einstein’s Postulate of Quantized Radiation” of this Chapter

It turns out that the procedure followed by these two approaches (leading to an expression for $I(\omega)$) have survived the new wave of quantum concepts; that is, they are still considered valid. **What has changed then?** What has changed dramatically, due to the advent of the quantum ideas, is the specific way to calculate the average-energy of an atom (in approach-1 method) or the average energy of an electromagnetic mode (approach-2 method.) A detailed description of these quantum modifications incurred in the calculation of $I(\omega)$ constitutes the essence of Chapter 3.

In what follows, we will mainly concentrate on a detailed account of the approach-1 referred above. If you want to skip the details, you can jump to expression (32) below.

**3.1.B.a Emission of radiation by an accelerated charge**

Electromagnetic radiation is emitted by charges in accelerated motion.

---

**Electric field $E$ produced by an accelerated charge.** At a point “P”, $E$ is oriented perpendicular to the line of sight and its magnitude proportional to the component of the acceleration-vector along the direction perpendicular to the line of sight.
Fig. 3.8 An electron in an atom modeled as a charge attached to the end of a spring of appropriate spring constant of natural frequency $\omega_0$.

$$E(r,t) = -\frac{q_e \sin(\theta)}{4\pi\varepsilon_0 c^2} \frac{1}{r} a(t - r/c)$$

Electric field produced by an accelerated charge $q_e$

Retardation time

The Poynting vector $\vec{S}$

$$\vec{S} = \varepsilon_0 E^2 \vec{c}$$

Its magnitude $S$ gives the amount of energy per square meter per unit time that passes through a surface that is normal to the direction of propagation.

Radiation power $P$ emitted by an accelerated charge

$$P = \int S \ dA = \int \left[\varepsilon_0 c q_e^2 \sin^2(\theta) \frac{1}{r^2} a'^2 \right] dA$$

$$P = \frac{q_e^2}{6\pi\varepsilon_0 c^3} a'^2$$

where $a' \equiv a(t') = a(t - r/c)$

Emission of radiation by a charge under harmonic oscillation

Consider that a charge $q_e$ oscillates with angular frequency $\omega$.
\[ x(t) = \text{Real} \{ x_0 e^{j \omega t} \} = x_0 \cos(\omega t) \]  

(12)

Here, \( \omega \) is not necessarily the natural frequency \( \omega_0 \) of the oscillator (see Fig 3.8). For example, the oscillator may be being driven by an external source of arbitrary frequency \( \omega \).

In what follows we will be evaluating time averages of physical quantities (let’s call it \( g \)) over one period \( T = 2\pi / \omega \) of oscillation:

\[ < g > = \frac{1}{T} \int_0^T g(t) \, dt \]  

(13)

Notice, for example, that the time average of \( \cos^2(\omega t) \) over one period is ½.

We also will be using indistinctly \( \cos(\omega t) \) or \( e^{j \omega t} \). For the latter it is assumed that we will take the real part after the calculation in order to obtain a proper (classical) physical interpretation.

From expression (11) we calculate an acceleration \( a' = -\omega^2 x_0 e^{j \omega t} \) and

\[ < a'^2 > = (1/2) \omega^4 |x_0|^2. \]

Replacing these values in expression (11) gives,

\[ < P > = \frac{q_e^2 |x_0|^2}{12\pi \varepsilon_0 c^3} \omega^4 \]

Total average power emitted by a charge \( q_e \) undergoing harmonic oscillations with amplitude \( x_0 \), and angular frequency \( \omega \).

(14)

3.1.B.b Light Scattering

Consider a plane wave (filling all the space) traveling in a specific direction, let’s say +X. Here we highlight only the temporal variation:

\[ E(t) = \text{Real} \{ E_0 e^{j \omega t} \} = E_0 \cos(\omega t) \]  

(15)

time variation of the electric field of the incident radiation.

A traveling wave carries energy. A quantification of this travelling energy is expressed by the corresponding Poynting vector \( \mathbf{S} = \varepsilon_0 E^2 \mathbf{c} \); its average value, according to (12), is equal to,
\[
<S> = \frac{1}{2} \varepsilon_o c E_o^2
\]

\(<S>\) is the average energy per square meter per unit time passing through a surface normal to the direction of propagation.

\(<S>\) is also called the intensity "I" 

\[
E = E_o e^{j \omega t}
\]

\(E_o\) is the amplitude of the electric field of the incident wave

**Fig. 3.10** Incident plane-wave radiation of frequency \(\omega\).

When an atom characterized by a resonance frequency \(\omega_o\), is placed in a region where there is a bath of electromagnetic radiation, the radiation’s electric field \(E = E_o e^{j \omega t}\) will drive the atom’s charge \(q_e\) up and down; that is, it will accelerate the charge thus causing the atom to re-emit electromagnetic radiation. This process, which occurs at any frequency \(\omega\), is called *scattering*. That is,

**scattering is the process by which energy is absorbed by an atom from the incident radiation field and re-emitted in all directions.**

**Fig. 3.11** Incident light is absorbed and (re-emitted) scattered by an atom in all directions.
Let’s calculate the energy absorbed (hence re-emitted) by the charge

Actually the energy emitted by the accelerated charge has already been calculated in expression (14) above, except that we need to find out the amplitude of oscillation $x_0$; the latter will depend on the electric field amplitude $E_o$, as well as the frequency $\omega$ of the incident radiation. In other words, let’s calculate the relationship between $E_o$, $\omega$, and $x_0$.

![Diagram of atom as an oscillator](image)

**Fig. 3.12** Atom modeled as an oscillator of natural frequency $\omega_o$. The ability of the oscillator to absorb energy from the incident radiation depends on $\omega$.

To find $x_0$, let’s model the atom as a damped harmonic oscillator. Accordingly the equation of motion for the charge $q_e$ is given by,

$$
\frac{m_e}{2} \frac{d^2 x}{dt^2} + m_e \gamma \frac{dx}{dt} + kx = q_e E_o e^{j\omega t}
$$

Here the term $m_e \gamma \frac{dx}{dt}$ accounts for the presence of a dissipation energy source, which, in our case, can be identified in the loss of energy due to the electromagnetic radiation by the accelerated charge.

A stationary solution of (18) is given by,

$$
x = [x_0 e^{j\phi}] e^{j\omega t}
$$

where
\[ x_o = x_o(\omega) = \frac{(q_e / m_e)E_o}{\left( (\omega_o^2 - \omega^2)^2 + \gamma^2\omega^2 \right)^{1/2}} \quad \text{Amplitude of oscillation as a function of frequency} \quad (20) \]

and

\[ \varphi = \tan^{-1}\left( -\frac{\gamma\omega}{\omega_o^2 - \omega^2} \right) \quad (20)' \]

Expression (20) indicates that the amplitude of oscillation \( x_o \) (and hence the acceleration) of the charge depends on the incident radiation’s frequency \( \omega \).

Let’s proceed now to calculate the total power radiate by the accelerated charge under the influence of an electric filed of amplitude \( E_0 \) and frequency \( \omega \). Replacing the value of \( x_o \) given in (20) for \( x_o \) into the expression for the radiation power \( < P > = \frac{q_e^2 x_o^2}{12\pi\epsilon_0 c^3} \omega^4 \) given in (14), one obtains, \( < P > = \frac{q_e^2}{12\pi\epsilon_0 c^3} \omega^4 \frac{(q_e / m_e)^2 E_o^2}{(\omega_o^2 - \omega^2)^2 + (\gamma\omega)^2} \). Rearranging terms,

\[ < P > = \left[ \frac{1}{2} \epsilon_o c E_o^2 \right] \frac{8\pi}{3} \left( \frac{q_e^2}{4\pi\epsilon_0 m_e c^2} \right)^2 \frac{\omega^4}{(\omega_o^2 - \omega^2)^2 + (\gamma\omega)^2} = < P >(\omega) \quad (21) \]

Expression (21) gives the total average energy emitted by the charge \( q_e \) when subjected to a harmonic electric field (given in expression (15) ) of amplitude \( E_o \) and frequency \( \omega \).

Notice the expression \( \frac{1}{2} \epsilon_o c E_o^2 \) (incident energy per unit area per second, i.e. the incident intensity \( I_o \)) has been factored out in expression (21). This is convenient, for it allows to interpret (21) the following way: Out of the incident intensity \( I_o \) present in the cavity, a ‘fraction’ of it equal to \( \frac{8\pi}{3} \left( \frac{q_e^2}{4\pi\epsilon_0 m_e c^2} \right)^2 \frac{\omega^4}{(\omega_o^2 - \omega^2)^2 + (\gamma\omega)^2} \) is present in the form of scattered power. We say ‘fraction’ because the units of that last expression is area (not a simple fraction number). Hence, it is better to interpret (21) in terms of “scattering cross section”.

Note (dated 09-2012).

Expression (21) quantifies the amount of energy that the the atom is able to re-radiate (due to the fact that is is a charge) upon the incidence of an harmonic electric field of amplitude \( E_o \) and frequency \( \omega \).
It has nothing to do with the ability of the atom to capture energy from radiation in the cavity (as the concept of scattering cross section may erroneously suggest). Hence, be careful with the proper interpretation of the “scattering cross section concept.”

The concept of scattering cross section

If we considered a hypothetical cross section of area $\sigma$ intersecting the incident radiation, the amount of energy per second hitting that area would be

$$< P > = \left[ \frac{1}{2} \varepsilon_o c E_o^2 \right] \sigma = I \sigma \quad (22)$$

$$< S > = \frac{1}{2} \varepsilon_o c E_o^2$$

$< S >$ is called the light intensity $I$

---

**Fig. 3.13** Pictorial representation of scattering cross section. Notice, this has nothing to do with the size of the atoms nor the spatial distribution of atoms inside the cavity. It is just a measurement of the ability of the atom (once radiation impinges on it) to radiate energy in all directions.

One can use the analogy of an affective area being intercepted by the incident radiation to define how effectively the radiation is absorbed and scattered (i.e. re-emitted) by an atom. In effect, comparing expressions (21) and (22), the total power scattered by an atom is numerically equal to the energy per second incident on a “surface” of cross-section area $\sigma_{scattering}$,

$$< P >_{scattering} = \left[ \frac{1}{2} \varepsilon_o c E_o^2 \right] \sigma_{scattering} \quad (23)$$

where

$$\sigma_{scattering} = \frac{8 \pi}{3} \left( \frac{q_e^2}{4 \pi \varepsilon_o m_e c^2} \right)^2 \frac{\omega^4}{\left( \omega_o^2 - \omega^2 \right)^2 + (\gamma \omega)^2} \quad (24)$$

$\sigma_{scattering}$ has units of area.
Note (Dated 09-2012) $\sigma_{\text{scattering}}$ is an indicator of the ability of the atom scatter light once it is excited by an harmonic electric field of amplitude $E_o$ and frequency $\omega$. We cannot ask, we cannot expect, the atom to scatter more (or less) than $I_o \sigma$ (where $I_o$ is the intensity associated to the filed incident on the atom).

Expression (24) indicates that the closer $\omega$ is to $\omega_o$, the higher the re-emitted energy.

3.1.B.c Electromagnetic Radiation Damping: What is the value of $\gamma$?

We address here the fact that

- the term $m_e \gamma \frac{dx}{dt}$ in Eq. (18) (the term in the differential equation that takes into account the energy dissipation,)
- should be compatible with expression (21) (that gives the electromagnetic energy dissipated by the oscillator.)

We should require then that these two expressions be consistent with each other.

Indeed,

- On one hand, the power dissipate by a oscillator is given by
  \[
  [\text{force}] \times [\text{velocity}] = \left[ m_e \gamma \frac{dx}{dt}, \left( \frac{dx}{dt} \right) \right] = \left[ m_e \gamma (j \omega x) \right] (j \omega x) = -m_e \gamma \omega^2 x^2. \]
  Here we use the expression for $x(t)$ given in (19) $x = x_o e^{j \omega t} e^{j \omega t}$.

The average value of the dissipated power will be,

\[
-(1/2)m_e \gamma \omega^2 x_o^2.
\]
On the other hand, according to (14), the emitted electromagnetic power is,

\[ < P > = \frac{q^2 x_o^2}{12\pi\varepsilon_o c^3} \omega^4 \]

The last two expressions should be equal.

\[ (1/2) m \gamma \omega^2 x_o^2 = \frac{q^2 x_o^2}{12\pi\varepsilon_o c^3} \omega^4 \]

This allows to identify \( \gamma = \frac{q^2}{6\pi\varepsilon_o m c^3} \omega^2 \). Rearranging terms,

\[ \gamma = \frac{2}{3e} \frac{q_e^2}{4\pi\varepsilon_o m_e c^2} \omega^2 \]

\[ \text{electromagnetic radiation damping constant} \]

For practical purposes, however (given the very narrow bandwidth of the cross section \( \sigma(\omega) \) shown in Fig 3.10 above,) \( \gamma \) will be typically end up being evaluated at \( \omega = \omega_0 \)(i.e. the narrow bandwidth of \( \sigma(\omega) \) tell us that most of the physics happens around \( \omega = \omega_0 \).)

**Rate at which the oscillator looses energy**

(A more detailed description of this section is given in the supplementary Appendix-3 of this chapter.)

Let

\[ W = W(t) \text{ be the average energy of an oscillating charge at a given time } t. \]

If the oscillating charge is left alone to oscillate, its amplitude of vibration will die out progressively as the oscillator looses its mechanical energy by emitting electromagnetic radiation.

If the motion of the oscillator is alternatively modeled by a mechanics equation of motion \( m_e \frac{d^2x}{dt^2} + m_e \gamma \frac{dx}{dt} + kx = q_e E_o e^{j\omega t} \), it can be calculated that the rate at which the oscillating charge looses energy is given by,

\[ \frac{dW}{dt} = -\gamma W \]

with its corresponding solution

\[ W(t) = W_0 e^{-\gamma t} \]
As an example, an atom that has a resonance frequency corresponding to $\lambda = 600$ nm, would have a damping constant of $\gamma \sim 10^8$ s$^{-1}$. That is, the radiation will effectively dye out after $\sim 10^{-8}$ s (or after $\sim 10^7$ oscillations.)

3.1.C Radiation and thermal equilibrium

Let’s consider an atom enclosed in a box made of mirror walls which contains electromagnetic radiation. Radiation re-emitted by the atom remains inside the box undergoing multiple bounces on the mirror walls. Let’s further assume that the temperature of the whole system is $T$.

How to make the temperature $T$ intervene in an expression like (14) that gives the power scattered by an atom in the form $\langle P \rangle_{\text{scattering}} = \frac{q_e^2 |x_o|^2}{12 \pi \varepsilon_o c^3} \omega^4$?

It is plausible to assume that the equilibrium temperature should correspond to proper value of the amplitude of the electric field, $E_o$, since the higher the value of $E_o$, the higher the charge’s amplitude of vibration $x_o$, the greater temperature to be associated with the atom (i.e. the amplitude of the oscillator should increase with temperature.)

If our assumption were correct, how to find then the proper value of $E_o$ corresponding to a given temperature $T$?

Aiming to find a proper answer let’s outline some considerations:

- If an atomic oscillator had no charge, it would oscillate forever. It would have an average energy $W$ compatible with the temperature in the box; that is
\[ W = W(T) \]. In other words, it would oscillate forever with an amplitude settled by the temperature \( T \) in the box. \( \frac{1}{2} k x_o^2 = \frac{1}{2} kT \) or \( \frac{1}{2} m \omega_0^2 x_o^2 = \frac{1}{2} kT \).

- But our atomic oscillator is charged. If it were left alone, its amplitude of vibration \( x_o \) would die out progressively, as the oscillator looses its energy by emitting electromagnetic radiation.

- If our atomic charged oscillator were in physical contact with other atoms, energy in the proper amount will be supplied by their mutual collisions as to keep the same temperature among themselves. Here, however, we will consider that the energy is supplied via electromagnetic interaction: the atom draws energy from the radiation existent in the cavity to compensate the energy being lost by radiation (accelerated charged particles emit radiation.) When this compensation of energy matches, then we are at an equilibrium situation, which inherently should occur at a given temperature (the latter settling the charge's amplitude of oscillation at a corresponding value.) We will retake the subject of temperature dependence of \( x_o \) in Section 3.1.c.b below.

### 3.1.C.a Light intensity spectral density \( I(\omega) \) at equilibrium

To formalize the equilibrium situation we have to keep in mind that radiation of different frequencies might be present in the cavity. It is convenient, then, to introduce the concept of spectral density \( I(\omega) \):

- \( I(\omega) = \) light intensity spectral density
- \( I(\omega) d\omega = \) light intensity of frequency \( \omega \) within a range \( \omega, \omega + d\omega \) inside the box
- contribution to the average electromagnetic energy per square meter per unit time passing through a surface normal to the direction of propagation from radiation components of frequency \( \omega \) within a range \( d\omega \)

Units of \([ I(\omega) ] = \frac{\text{Intensity}}{\Delta \omega} \) = \( \frac{J}{(m^2 s)} = \frac{J}{m^2} \)

Before establishing the condition of equilibrium, let's make two pertinent observations:

- According to (21) and (22), the total power that the atom is able to emit (scatter) is given by,

\[
\int_{0}^{\infty} P > (\omega) d\omega = \int_{0}^{\infty} I(\omega) \sigma(\omega) d\omega,
\]
It is an integral because the atom is exposed to all the frequencies \( \omega \) existent in the cavity. However, the maximum emission of power occurs when \( \omega \) is close to to \( \omega_0 \) (because \( \sigma(\omega) \) has a sharp peak at \( \omega_0 \)).

- On the other hand, this same amount of emitted power (i.e. \( \int_0^\infty I(\omega)\sigma(\omega)d\omega \)), can be seen from the perspective of a system loosing energy due to a damping process \( \frac{d[W(\omega)]}{dt} = -\gamma(\omega)[W(\omega)] \) characterized by a damping constant \( \gamma(\omega) \). On the other hand, the requirement of compatibility between \( i) \) power re-radiated by the atom, and \( ii) \) a simple damping harmonic oscillator model

\[
m_e \frac{d^2x}{dt^2} + m_e \gamma \frac{dx}{dt} + kx = q_e E_o e^{j\omega t} ,
\]

lead to expression \( (25) \)

\[
\gamma(\omega) = \frac{2}{3c} \frac{q_e^2}{4\pi\varepsilon_0 m_e c^2} \omega^2 .
\]

But since all the dynamics occurs at \( \omega=\omega_0 \), that is \( W(\omega) \sim 0 \) for \( \omega \neq \omega_0 \), we can use \( \frac{dW}{dt} = -\gamma(\omega_0) W \), with the interpretation that \( W \) is the total energy of the atom.

**Formalization of the thermal equilibrium condition:**

How much light intensity spectral density \( I(\omega) \) there must be inside the box at temperature \( T \) for,

- the electromagnetic energy re-emitted by the oscillator (which should come from the radiation bath in the cavity) per unit time

\[
\frac{dW}{dt} = \int_0^\infty <P>(\omega) d\omega = \int_0^\infty I(\omega)\sigma(\omega)d\omega \quad \text{(29)}
\]

to be equal to

- the energy lost by the oscillator per unit time

\[
\frac{dW}{dt} = -\gamma W \quad \text{(30)}
\]

**Average energy of the oscillator at temperature \( T \)**
Let’s evaluate the integral that appears in (29).

- Since the expression for $\sigma(\omega)$ peaks at $\omega \approx \omega_0$, then extending the integral down to $\omega = -\infty$ does not cause any significant change (this is done just to facilitate the calculation)

$$
\int_{-\infty}^{\infty} I(\omega)\sigma(\omega)d\omega = \int_{-\infty}^{\infty} I(\omega)\sigma(\omega)d\omega
$$

where $\sigma_{scattering} = \frac{8\pi}{3} \left( \frac{q_e^2}{4\pi\varepsilon_0 m_e c^2} \right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$

- For the same reason that only the values of $\omega$ very close to $\omega_0$ will significantly contribute to the integral we can picture in our mind that

$$
\int_{-\infty}^{\infty} I(\omega)\sigma(\omega)d\omega \approx \int_{\omega_0 - \gamma}^{\omega_0 + \gamma} I(\omega)\sigma(\omega)d\omega
$$
Accordingly the following approximations can be considered appropriate,

\[ \omega^2 - \omega_o^2 = (\omega + \omega_o)(\omega - \omega_o) \approx (2\omega_o)(\omega - \omega_o) \]

\[ \frac{\omega^4}{(\omega_o^2 - \omega^2)^2 + (\gamma \omega)^2} \approx \frac{\omega_o^4}{[2\omega_o(\omega - \omega_o)]^2 + (\gamma \omega_o)^2} \]

\[ \approx \frac{\omega_o^2}{4(\omega - \omega_o)^2 + \gamma^2} \]

\[ I(\omega) \approx I(\omega_o) \]

All these approximations lead to

\[
\int_0^{\infty} I(\omega)\sigma(\omega)d\omega \approx \int_0^{\infty} I(\omega)\sigma(\omega)d\omega
\]

\[
= \int_{-\infty}^{\infty} I(\omega) \left[ \frac{8\pi}{3} \left( \frac{q_e^2}{4\pi\varepsilon_o m_e c^2} \right)^2 \frac{\omega_o^4}{(\omega_o^2 - \omega^2)^2 + (\gamma \omega)^2} \right] d\omega
\]

\[
= \int_{-\infty}^{\infty} I(\omega) \left[ \frac{8\pi}{3} \left( \frac{q_e^2}{4\pi\varepsilon_o m_e c^2} \right)^2 \frac{\omega_o^2}{4(\omega - \omega_o)^2 + \gamma^2} \right] d\omega
\]

\[
= I(\omega_o) \frac{2\pi}{3} \left( \frac{q_e^2}{4\pi\varepsilon_o m_e c^2} \right)^2 \omega_o^2 \int_{-\infty}^{\infty} \left[ \frac{1}{(\omega - \omega_o)^2 + (\gamma/2)^2} \right] d\omega
\]

(\text{using } \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a})

\[
= I(\omega_o) \frac{2\pi}{3} \left( \frac{q_e^2}{4\pi\varepsilon_o m_e c^2} \right)^2 \omega_o^2 \frac{1}{\gamma/2} \left( \frac{\pi}{2} - (-\frac{\pi}{2}) \right)
\]

\[
= I(\omega_o) \frac{2\pi}{3} \left( \frac{q_e^2}{4\pi\varepsilon_o m_e c^2} \right)^2 \omega_o^2 \frac{2\pi}{\gamma}
\]

\[
\int_0^{\infty} I(\omega)\sigma(\omega)d\omega = I(\omega_o) \frac{(2\pi)^2}{3\gamma} \left( \frac{q_e^2}{4\pi\varepsilon_o m_e c^2} \right)^2 \omega_o^2
\]

(31)
At equilibrium we should have, \[ \frac{dW}{dt} = \int_{0}^{\infty} I(\omega)\sigma(\omega)d\omega = \left| \frac{dW}{dt} \right| = \gamma \ W, \] which leads to

\[ I(\omega_o) \left( \frac{2\pi}{3\gamma} \right)^2 \left( \frac{q_e^2}{4\pi\epsilon_0 m_e c^2} \right)^2 \omega_o^2 = \gamma W \]

\[ I(\omega_o) = \frac{3}{(2\pi)^2} \left( \frac{4\pi\epsilon_0 m_e c^2}{q_e^2} \right)^2 \frac{\gamma^2}{\omega_o^2} \frac{\omega_o^2}{W} \]

Using expression (25) for the value of \( \gamma = \frac{2}{3c} \frac{q^2}{4\epsilon_0 m_e c^2} \omega^2 \) evaluated at \( \omega = \omega_o \), one obtains,

\[ I(\omega_o) = \frac{3}{(2\pi)^2} \left( \frac{2}{3c} \right)^2 \omega_o^2 W, \]

\[ I(\omega_o) = \frac{1}{3\pi^2 c^2} \omega_o^2 \frac{W}{W} \]  \hspace{1cm} (32)

\( I(\omega_o) \) is the light spectral density at \( \omega = \omega_o \).

Here \( \omega_o \) is the natural frequency of the oscillator we were focusing in. Had we used an oscillator of a different natural frequency, let’s say \( \omega_o’ \), we would have obtained a similar expression (32) but with \( \omega_o’ \) instead of \( \omega_o \). Hence, in general,

\[ I(\omega) = \frac{1}{3\pi^2 c^2} \omega^2 W \]  \hspace{1cm} (32)’

Required light intensity spectral density \( I(\omega) \) inside the box at temperature \( T \), in order to maintain equilibrium inside the cavity.
Notice

Units of \([I(\omega) \ d\omega]\) = [Intensity] = \(\frac{J}{m^2 \ s}\)

Units of \([I(\omega)]\) = \(\frac{[Intensity]}{[\Delta \omega]}\) = \(\frac{J/(m^2 \ s)}{1/s} = \frac{J}{m^2}\)

It is worth to highlight that,

- Expression (32)' has remained undisputed. That is, it is still considered correct even when the new quantum mechanics concepts are introduced.
- It is in the calculation of the average energy \(W\) where the classical and quantum approaches fundamentally diverge.

### 3.1.C.b Classical calculation of the atom’s average energy \(W\).

In classical statistical mechanics there exists a very general result so called “equipartition theorem,” which states that the mean value of a quadratic term in the energy is equal to \(\frac{1}{2} k_b T\). Here \(k_b\) is the Boltzmann’s constant and \(T\) is the absolute temperature.

**The Boltzmann distribution**

The equipartition theorem can be obtained from the Boltzmann’s probability distribution for a small system \(\mathcal{A}\) in equilibrium with a (huge) reservoir at temperature \(T\). The Boltzmann distribution states that the probability that the system \(S\) be found in a state of energy \(E\) is proportional to \(e^{-E/k_b T}\); that is,

\[
P(E) \propto e^{-E/k_b T}
\]

\[
= Ce^{-E/k_b T} \quad \text{probability to find the system} \quad \mathcal{A} \quad \text{in a state of energy} \quad E
\]
Fig 3.18 Left: A system interacting with a thermal reservoir. Right: Boltzmann’s distribution to find the system in a state of energy $E$.

The values of $E$ could go from 0 to infinity (the reservoir being in charge of keeping the temperature constant); but, as the expression above indicates, the states of lower energy have a higher probability.

Since for a given energy there may be several states characterized by the same energy, it is usual to define,

$$g(E)dE \quad \text{number of states with energy}$$

$$E, \text{ within an interval } dE,$$

thus giving

$$Ce^{-E/k_BT}g(E)dE \quad \text{probability to find the system } \mathcal{A} \text{ in a state of energy between } E \text{ and } E+ dE,$$

which suggests to rather identify a probability-density $P(E)$ defined as follows,

$$P(E)dE = Ce^{-E/k_BT}g(E)dE \quad \text{probability to find the system in a state of energy between } E \text{ and } E+ dE,$$

with $C$ being a constant to be determined.

Since the probabilities added over all the possible states should be equal to 1, we must require,

$$\int_0^\infty Ce^{-E/k_BT}g(E')dE' = 1,$$

which gives,

$$C^{-1} = \int_0^\infty e^{-E/k_BT}g(E')dE'$$

A self-consistent expression for $P(E)$ is therefore given by,

$$P(E)dE = \frac{e^{-E/k_BT}g(E)dE}{\int_0^\infty e^{-E'/k_BT}g(E')dE'}$$

(Notice in the denominator we are using a ‘dummy’ variable $E'$.)

From expression (37) we can formally calculate the average energy of the system,
\begin{equation}
\langle E \rangle = \frac{\int_{0}^{\infty} E e^{-E/k_B T} g(E)dE}{\int_{0}^{\infty} e^{-E/k_B T} g(E')dE'}
\end{equation}

**The Equipartition Theorem**

It turns out, very often the energy of the system may contain a quadratic term. Consider for example

\[ E = \frac{p_x^2}{2m} + \frac{1}{2} kx^2 + \ldots \]

and we would like to calculate, for example, the average value of the kinetic energy term \( \frac{p_x^2}{2m} \) alone. As we know, being the system in contact with a heat reservoir, the value of \( \frac{p_x^2}{2m} \) is sometimes high, sometimes it is low because it gains or loses energy from the heat reservoir; we would like to know what would be its average value

\[ \left\langle \frac{p_x^2}{2m} \right\rangle. \]

\begin{equation}
\left\langle \frac{p_x^2}{2m} \right\rangle = \frac{\int_{0}^{\infty} p_x^2 e^{-p_x^2/k_B T} dp_x}{\int_{0}^{\infty} e^{-p_x^2/k_B T} dp_x}
\end{equation}

Let's call

\[ \beta = \frac{1}{k_B T} \]

In terms of \( \beta \) expression (39) becomes,

\begin{align}
\left\langle \frac{p_x^2}{2m} \right\rangle &= \int_{0}^{\infty} \frac{p_x^2}{2m} e^{-\beta p_x^2} dp_x = \int_{0}^{\infty} \frac{\partial}{\partial \beta} e^{-\beta p_x^2} dp_x = -\frac{\partial}{\partial \beta} \int_{0}^{\infty} e^{-\beta p_x^2} dp_x \\
&= \int_{0}^{\infty} e^{-\beta p_x^2} dp_x - \int_{0}^{\infty} e^{-\beta p_x^2} dp_x
\end{align}

\begin{equation}
\left\langle \frac{p_x^2}{2m} \right\rangle = -\frac{\partial}{\partial \beta} \ln \left[ \int_{0}^{\infty} e^{-\beta p_x^2} dp_x \right]
\end{equation}
Defining the variable $u = \frac{\beta}{\sqrt{2m}} p_x$,

\[
\left< \frac{p_x^2}{2m} \right> = -\frac{\partial}{\partial \beta} \ln \left[ \sqrt{\frac{2m}{\beta}} \int_0^\infty e^{-u^2} \, du \right] = -\frac{\partial}{\partial \beta} \ln \left[ \frac{1}{\beta} \sqrt{2m} \int_0^\infty e^{-u^2} \, du \right]
\]

\[
\left< \frac{p_x^2}{2m} \right> = -\frac{\partial}{\partial \beta} \ln \left( \frac{1}{\sqrt{\beta}} \right) + \ln \left( \sqrt{2m} \int_0^\infty e^{-u^2} \, du \right)
\]

\[
\frac{\partial}{\partial \beta} \frac{1}{2} \ln \beta = \frac{1}{2\beta}
\]

\[
\left< \frac{p_x^2}{2m} \right> = \frac{1}{2} k_B T
\]

\[
(41')
\]

Had we chosen any other quadratic term of the energy we would have obtained the same result. This is the equipartition theorem. It states,

If the energy $W$ of the system has the form

\[
E = \frac{p_x^2}{2m} + \frac{l}{2} k_B x^2 + \ldots
\]

the average value of each independent quadratic term is equal to $\frac{1}{2} k_B T$.

\[
W = < E > = \frac{1}{2} k_B T + \frac{1}{2} k_B T + \ldots
\]

The ultraviolet catastrophe

Let’s assume there are $f$ different quadratic terms in the expression for the total energy $W$ (translation motion, rotational motion, ..., etc.). The equipartition theorem leads to $W = f \frac{l}{2} k_B T$. Using this result in expression (32)' $I(\omega) = \frac{l}{3\pi^2 c^5} \omega^3 W$, one obtains

\[
I(\omega) = \frac{l}{3\pi^2 c^5} \omega^3 f \frac{l}{2} k_B T, \quad \text{or,}
\]

\[
I(\omega) = \frac{f \cdot k_B T}{6\pi^2 c^5} \omega^3 \quad \text{classical prediction}
\]

(43)
The serious discrepancy between the experimental results and the theoretical prediction is called the ultraviolet catastrophe.

### 3.1.D The birthday of Quantum Physics: Planck’s Hypothesis to calculate the atom’s average energy \( W \)

To bring the theoretical prediction closer to the experimental results Planck considered the possibility of a violation of the law of equipartition of energy described above, expression (42).

The starting expression would still be expression (32) \( I(\omega) \frac{1}{3\pi^2c^2} \omega^2 W \), BUT with the average energy of the oscillator \( W \) not being constant (as the equipartition theorem predicts) but rather being a function of the frequency, \( W(\omega) \), with the following requirements,

\[
\omega^2 W(\omega) \xrightarrow{\omega \to 0} 0
\]

and

\[
\omega^2 W(\omega) \xrightarrow{\omega \to \infty} 0
\]

For the statistical calculation of \( W \),

\[
\text{Planck did not question the classical Boltzmann’s Statistics described in the section above; that Theory would still be considered valid.}
\]

Planck realized that he could obtain the desired behavior expressed in (44) if,

- rather than treating the energy of the oscillator as a continuous variable,
  - the energy states of the oscillator should take only discrete step values:
    \[
    0, \varepsilon, 2\varepsilon, 3\varepsilon, ...
    \]
  - the energy steps would be different for each frequency
\( \varepsilon = \varepsilon(\omega) \) \hfill (47)

where the specific dependence of \( \varepsilon \) in terms of \( \omega \) to be determined

Planck postulated that the energy of the oscillator is quantized

\[ \text{Fig. 3.20 An atom receiving radiation of frequency } \omega, \text{ can be excited only by discrete values of energy } 0, \varepsilon, 2\varepsilon, 3\varepsilon, \ldots \]

According to Planck, in the classical integral expression (37)

\[
W_{\text{classical}} = \langle E \rangle = \int_0^\infty E e^{-E/k_BT} g(E) dE = \int_0^\infty e^{-E/k_BT} g(E') dE', \]

one would have to replace:

\[ g(E) dE = 1 \quad [g(E) dE \text{ gives the number of states with energy } E \text{ within an interval } dE] \]

\[ dE \rightarrow \varepsilon \]

\[ \int_0^\infty dE \rightarrow \sum_{n=0}^{\infty} \]

thus obtaining,

\[
W_{\text{Planck}}(\omega) = \langle E \rangle = \frac{\sum_{n=0}^{\infty} E_n e^{-E_n/k_BT}}{\sum_{n=0}^{\infty} e^{-E_n/k_BT}} \hfill (48)
\]

where \( E_n = n \varepsilon(\omega) ; \quad n = 1, 2, 3, \ldots \)

A graphic illustration can help understand why this hypothesis could indeed work:

- First we show how classical physics evaluate the average energy.

\[
\langle E \rangle_{\text{classical}} = \int_0^\infty E [P(E)] dE = \text{area under the curve of } E P(E) \text{ vs } E
\]
Using Plank’s hypothesis \( \langle E \rangle_{Planck} = \sum_{n=0}^{\infty} E_n P(E_n) \)

**Case: Low frequency values of \( \omega \)**

For this case, Planck assumed \( \varepsilon \) should have a small value (for the reasons explained in Fig. 3.17).

\[
\varepsilon \approx \text{small value} \quad \text{(for small values of } \omega \text{)}
\]  

Indeed, comparing Fig. 3.21 and Fig. 3.22 one notices that if \( \varepsilon \) is small then the value of \( \sum_{n=0}^{\infty} E_n P(E_n) \) will be very close to the classical value.

It is indeed desirable that Planck’s results agree with the classic results at low frequencies, since the classical predictions and the experimental results agree well at low frequencies (see Fig. 3.19 above.)

**Case: High frequency values of \( \omega \)**
For this case, Planck assumed, Planck assumed $\varepsilon$ should have a bigger value

$$\text{Step } \varepsilon \approx \text{a big value} \quad \text{(for high values of } \omega) \quad (50)$$

$$\sum_{n=0}^{\infty} E_n P(E_n)$$

Fig. 3.23 Schematic representations to calculate the average energy of the oscillator, assuming the oscillator can admit only discrete values of energy, for the case where the separation $\varepsilon$ between contiguous energy levels being of relatively large value.

Notice in this case, the value of $\langle E \rangle_{\text{Planck}} = \sum_{n=0}^{\infty} E_n P(E_n)$ (depicted in Fig. 3.23) will be very different from the one depicted in Fig. 3.22 (the latter closer to the classical result.)

In fact, notice that $\sum_{n=0}^{\infty} E_n P(E_n)$ tends to zero as the step $\varepsilon$ is chosen larger.

This result is desirable, since the experimental results in Fig. 3.19 indicate that $\langle E \rangle$ tends to zero at high frequencies.

It appears then that the Planck’s model (48) could work. Now, which function $\varepsilon(\omega)$ could be chosen such that

$\varepsilon(\omega)$ is small at low $\omega$, and large at large $\omega$?

An obvious simple choice is to assume a linear relationship,

$$\varepsilon(\omega) = h \omega$$

where $h$ is a constant of proportionality to be determined.

Replacing (51) in (48) one obtains,

$$W_{\text{Planck}}(\omega) = \langle E \rangle = \frac{\sum_{n=0}^{\infty} E_n e^{-E_n/k_B T}}{\sum_{n=0}^{\infty} e^{-E_n/k_B T}} = \frac{\sum_{n=0}^{\infty} n h \omega e^{-n h \omega / k_B T}}{\sum_{n=0}^{\infty} e^{-n h \omega / k_B T}}$$

Let $\alpha \equiv \frac{h}{kT}$
\[ W_{\text{Planck}}(\omega) = \frac{kT}{\sum_{n=0}^{\infty} e^{-n\alpha \omega}} - kT \sum_{n=0}^{\infty} \frac{\partial}{\partial \alpha} e^{-n\alpha \omega} \sum_{n=0}^{\infty} e^{-n\alpha \omega} \]

Since \( d[\ln(u)]/dx = (1/u)du/dx \)

\[ W_{\text{Planck}}(\omega) = -kT \alpha \frac{\partial}{\partial \alpha} \ln[\sum_{n=0}^{\infty} e^{-n\alpha \omega}] \]

If we define \( x = e^{-\alpha \omega} \), then

\[ \sum_{n=0}^{\infty} e^{-n\alpha \omega} = 1 + x + x^2 + x^3 \ldots \]

We know that for \( x<1 \)

\[ 1 + x + x^2 + x^3 \ldots = \frac{1}{1-x} \]

\[ \sum_{n=0}^{\infty} e^{-n\alpha \omega} = \frac{1}{1-x} = \frac{1}{1-e^{-\alpha \omega}} \]

\[ W_{\text{Planck}}(\omega) = -kT \alpha \frac{\partial}{\partial \alpha} \ln[\sum_{n=0}^{\infty} e^{-n\alpha \omega}] = -kT \alpha \frac{\partial}{\partial \alpha} \ln[\frac{1}{1-e^{-\alpha \omega}}] \]

\[ = kT \alpha \frac{\partial}{\partial \alpha} \ln[1-e^{-\alpha \omega}] \]

\[ = kT \alpha \frac{\omega e^{-\alpha \omega}}{1-e^{-\alpha \omega}} = kT \alpha \frac{1}{e^{\alpha \omega} - 1} \]

\[ W_{\text{Planck}}(\omega) = \frac{1}{\hbar \omega} e^{\hbar \omega/kT} \]  \hspace{1cm} (52)

Notice: \( W_{\text{Planck}}(\omega) \xrightarrow{\omega \to 0} kT \)

\[ W_{\text{Planck}}(\omega) \xrightarrow{\omega \to \infty} 0 \]

With this result, expression (33) becomes

\[ I(\omega) = \frac{1}{3\pi^2 c^2} \omega^2 W_{\text{Planck}} \]  \hspace{1cm} (53)
Particle’s and wave’s energy quantization

- Historically, Planck initially (1900) postulated only that the energy of the oscillating particle (electrons in the walls of the blackbody) is quantized. The electromagnetic energy, once radiated, would spread as a continuous.
- It was not until later that Planck accepted that the oscillating electromagnetic waves were themselves quantized. The latter hypothesis was introduced by Einstein (1905) in the context of explaining the photoelectric effect, which was corroborated later by Millikan (1914).

3.1.B.g Light Energy Density $U(\omega)$ at equilibrium.

The Concept of Electromagnetic Modes
(see “Complement Chapter-3” notes (website))

3.2 PARTICLE-LIKE PROPERTIES of RADIATION

3.2.A Processes involving the absorption or scattering of radiation by particles

3.2.A.a Photoelectric effect Einstein’s hypothesis of the quantized radiation energy

Einstein’s interpretation:
- In the wave picture, the intensity of radiation is given by the average of the Poynting vector $S$,
  $$ I = <S> = \langle \epsilon_o c E^2 \rangle $$
  where $E$ is the electric field of the wave
In the *particle* picture, the intensity is written as

\[ I = N \ h \ \gamma \]

where \( N \) is the number of photons per unit time crossing a unit area perpendicular to the direction of propagation.

Einstein suggested that \( \left\langle E^2 \right\rangle \) could be interpreted as a measure of the average number of photons per unit volume.

- The radiation granularity concept leads to consider the light intensity as a statistical variable.
  - A light source emits photons randomly in all directions.
  - The exact number of photons crossing an area fluctuates (around an average number) in time and space.

### 3.2.A.b The Compton Effect

Scattering of a complete quantum of radiation in definite direction.

Compton explained the results on the assumptions that

- each electron scatters a complete quantum of radiation (photon); and
- the quanta of radiation are received from definite directions and are scattered in definite directions.

### 3.3 WAVE-LIKE PROPERTIES of PARTICLES

#### 3.3.A The Louis de Broglie Hypothesis

- The Compton’s Effect experiment had provided strong evidence on the *particle nature of radiation*. The typical *symmetry* expressed by *Nature* in many physical processes (for example, charges in motion produce magnetism and magnets in motion produce currents) may have led Louis de Broglie to propose the *wave nature of particles*.

- In 1924, Louis de Broglie, using arguments based on the properties of *wave packets*, proposed that,
the wave aspect of matter are related to its particle aspect in exactly the same quantitative way displayed by radiation.

\[ \lambda = \frac{h}{p} \]  

where \( \lambda \) is the wavelength of the wave associated with the particle’s motion; and

\[ \gamma = \frac{E}{h} \]  

the total energy \( E \) is related to the frequency \( \gamma \) of the wave associated with its motion.

Here \( h \) is the Planck’s constant; \( h = 6.6 \times 10^{-34} \text{ Js} \).

Since in the case of radiation (mass \( m = 0 \)) we already know that \( \lambda \gamma = c \), only one of the relationships above is needed to obtain both \( \lambda \) and \( \gamma \) from its energy and momentum’s particle properties.

In contrast, for a particle of mass \( m \neq 0 \), one needs both relationships given above to obtain the corresponding \( \lambda \) and \( \gamma \).

- **Order of magnitude calculation of \( \lambda_{\text{de Broglie}} \)**

  **Case:** \( m = 10^{-3} \text{ Kg}, \, v = 1 \text{ m/s} \)

  \[ p = mv = 10^{-3} \text{ Kg} \cdot \text{m/s} \]

  \[ \lambda = \frac{h}{p} = 6.6 \times 10^{-31} \text{ m}. \]

  This value is inaccessible to any measurement currently possible.

  **Case:** \( m_e = \text{electron’s mass} = 9.1 \times 10^{-31} \text{ Kg}, \, q_e = 1.6 \times 10^{-19} \text{ C} \)

  If the electron accelerated through a potential difference of \( V_o \) volts its kinetic energy will be determined by

  \[ q_e V_o = \frac{p^2}{2m} \]

  which gives

  \[ p = \left( 2m_e q_e V_o \right)^{1/2} \]

  For \( V_o = 50 \text{ Volts} \):

  \[ p = \left( 2m_e q_e V_o \right)^{1/2} = \left( 14.56 \times 10^{-48} \right)^{1/2} \]

  \[ p = 3.8 \times 10^{-24} \text{ m/s} \]

  \[ \lambda = \frac{h}{p} = \frac{6.6 \times 10^{-34} \text{ J} \cdot \text{s}}{(3.8 \times 10^{-24} \text{ m/s})} = 1.7 \times 10^{-10} \text{ m} \]

  This value is of the same order of magnitude of the X-rays’ wavelength.

3.3.B **Experimental confirmation of the de Broglie hypothesis.**

**Electron Diffraction**
In 1897 J. J. Thompson discovered the electron (establishing the ratio $q_e / m_e$) and was awarded the Nobel Prize in 1906.

The de Broglie wavelength $\lambda_{de\ Broglie}$ associated to an electron falls in the order of Angstroms (when accelerated at moderate voltages $\sim 50$ Volts, as shown in the example above), which happens to be the inter-atomic spacing in a crystalline solid.

In 1926 Elsasser suggested that the wave nature of particles could be tested by verifying whether or not electrons of the proper energy are diffracted by a crystalline solid. This idea was confirmed by Davisson and Germer (in the United States) and J. J. Thompson’s son G. P. Thompson in Scotland (1927).

**Schematic diagram of the Davisson-Germer experiment.**

**Fig. 3.19 Experimental setup for demonstrating the wave character of electrons**
If the incidents electrons were interpreted as waves of wavelength $\lambda$, reflecting from the Brag’s panes (similar to the x-rays), then they could give rise to interference phenomena. In particular, the condition for obtaining constructive interference (see figure above) is given by length of path ABC = n $\lambda$; or,

$$2d \cos (\theta/2) = n \lambda \quad (n=1,2,...)$$

From the figure above, $d = D \sin (\theta/2)$. Thus the condition of constructive interference becomes $2D \sin (\theta/2) \cos (\theta/2) = n \lambda$, or

$$D \sin (\theta) = n \lambda \quad (n=1,2,...) \quad (67)$$

For an accelerating voltage $V_o = 54$ volts, $\lambda_{de\ Broglie} = h/p = 1.66 \text{ Å}$ (as calculated above). According to expression (46) it would be expected to observe a maximum of interference at,

$$\sin (\theta) = \lambda/D = 1.66/2.15 = 0.77$$

$$\theta = 50.3 \text{ degrees} \quad (theoretical\ prediction.)$$

The figure below shows a schematic of the experimental result obtained by Davisson and Germer.

The peak signal of detected electrons around $\theta = 50^\circ$ supports the de Broglie hypothesis, since it can be explained as constructive interference of waves scattered by the periodic arrangement of the atoms in the crystal.

![Graph showing collector current I as a function of angle θ at V_o=54 Volts](image)

**Fig. 3.20** Experimental results obtained by Davisson and Germer. Collector current I as a function of the
detector’s angle (for a fixed value of the accelerating voltage)

- The electron interference is not between waves associated with one electron, and waves associated with another. Instead, it is the interference between different parts of the wave associated with a single electron.

The above can be demonstrated by using an electron gun of such low intensity that **one electron at a time incident the crystal** and by showing that the pattern of scattered electrons remains the same.

### 3.4 WAVE-PARTICLE DUALITY
- Not only electrons but all materials, charge or uncharged, show wavelike characteristics:
  - Estermann, Stern, and Frish
    Molecular beams of hydrogen and atomic beams of helium from a lithium fluoride crystal. The difficulties of these experiments lie in the low intensity of the molecular beams attainable.
  - Fermi, Marshall and Zinn
    Use the abundant slow neutrons available from nuclear reactors. (X-ray analysis is based on the photon-electron interaction; neutrons rather interact with the nucleus.)

- The particle aspect is emphasized when studying their emission and detection
- The wave aspect is emphasized when studying their motion

- The small value of \( h \) impedes the observation of wave-particle duality in the macroscopic world.
  Since \( \lambda = h / p \), a zero value for \( h \) would imply an absence of diffraction effects. On the other hand, the (relatively) large momentum of macroscopic objects compared to \( h \) also makes \( \lambda \) too small to be detected.
• The deeper understanding of the link between the wave model and the particle model is provided by a probabilistic interpretation of the wave-particle duality:
  
  ➢ In the case of **radiation**, it was Einstein who united the wave and particle theories (used in the explanation of the photoelectric effect.)
  
  ➢ Max Born (many years after) applied later similar arguments to unite the wave and particle theories of **matter**. (Einstein, however, did not accept it.)

  This latter aspect is more involved and becomes the subject of the next chapter.

Reference

Richard Feynman, “The Feynman Lectures on Physics,” Volume I, Chapter 32 (Sections 32-1 to 32-3) and Chapter 41 (Sections 41-2 and 41-3)
The electromagnetic damping constant

For comparison, let’s consider first the case of a dissipative force proportional to the velocity \( F_{\text{dis}} = -m \gamma \frac{dx}{dt} \).

The Eq. of motion is \[ m \frac{d^2x}{dt^2} + kx = F_{\text{ext}} \] which can be expressed in terms of energy balance as follows,

\[
\frac{d}{dt} \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} kx^2 \right] = P_{\text{me}}(t) - m\gamma \left( \frac{dx}{dt} \right)^2
\]  

For \( X = X_0 e^{i\omega t} \), the average dissipating power is given by,

\[
<P_{\text{me}} > = < F_{\text{me}} \frac{dx}{dt} > = < -m\gamma \frac{dx}{dt} \frac{dx}{dt} > = -m \gamma < \frac{dx}{dt} \frac{dx}{dt} >
\]

\[= -m\gamma \left[ \frac{1}{2} \left( \frac{dx}{dt} \right)^2 \right] = -\frac{1}{2} m\gamma \omega^2 |x_0|^2 \]

The average total mechanical energy of the oscillator varies as,

\[
\left\langle \frac{d}{dt} \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} kx^2 \right] \right\rangle = \left\langle P_{\text{me}}(t) \right\rangle = \frac{1}{2} m \gamma \omega^2 |x_0|^2
\]

For a charge \( q \) oscillating according to \( X = X_0 e^{i\omega t} \), the average electromagnetic energy dissipated is given by,

\[
\left\langle \frac{d}{dt} \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} kx^2 \right] \right\rangle = \left\langle P_{\text{em}}(t) \right\rangle = \frac{q^2 \omega^4}{12 \pi \varepsilon_0 c^3} |x_0|^2
\]

For the sake of comparison of the effective dissipation energy between the mechanical dissipation and electromagnetic dissipation cases, the following definition is typically introduced,

\[
\frac{1}{2} m \gamma_{\text{electromagnetic}} \omega^2 = \frac{q^2 \omega^4}{12 \pi \varepsilon_0 c^3}
\]

\[
\gamma_{\text{electromagnetic}} = \frac{q^2 \omega^2}{6 m \pi \varepsilon_0 c^3} = \frac{2 q^2 (4 \pi \varepsilon_0)}{3 c mc^2} \omega^2
\]

Thus

\[
\left\langle \frac{d}{dt} \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} kx^2 \right] \right\rangle = \left\langle P_{\text{em}}(t) \right\rangle = \frac{1}{2} m \gamma_{\text{electromagnetic}} \omega^2 |x_0|^2
\]

Note: When a external source is present and makes the charge oscillate at \( \omega \neq \omega_0 \), we have to use
\[
\left\langle \frac{d}{dt} \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 + \frac{1}{2} kx^2 \right] \right\rangle = \langle P_{\text{ext}}(t) \rangle - \frac{1}{2} m \gamma_{\text{electromagnetic}} \omega^2 |x_0|^2
\]

which is essentially,

\[
\frac{\Delta W}{\Delta t} = \langle P_{\text{ext}}(t) \rangle - \frac{1}{2} m \gamma_{\text{electromagnetic}} \omega^2 |x_0|^2
\]

Notice this term \( \frac{1}{2} m \omega^2 |x_0|^2 \) is smaller than \( W \), and since \( |x_0|^2 \) is also much smaller than the amplitude at resonance, therefore the loss of energy outside the resonance frequency is \textbf{comparatively very small}.

---

2 Richard Feynman, “The Feynman Lectures on Physics,” Volume I, Chapter 24 (Section 24-1 and 24-2).
3 Feynman Lectures, Vol - I, Chapter 32