CHAPTER-5

QUANTUM BEHAVIOR of PARTICLES and the HEISENBERG’S UNCERTAINTY PRINCIPLE

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CHAPTER-5
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In chapter 4 we used the coupling between i) the De Broglie wave-particle duality principle and ii) the wave-packet’s Fourier analysis, to illustrate the relationship between the spatial $\Delta x$ and linear momentum $\Delta p$ uncertainties. Such mathematical analysis led us to gain an understanding of the Heisenberg’s uncertainty principle. In this Chapter we emphasize again such an important principle but from a more physical content background. We consider the diffraction that electrons (or alternatively photons) undergo when passing through a slit, and compare them with the case of using classical bullets instead. The more striking points constitute a) the inability of the observer to discern the specific path followed by each diffracting particle, and b) that the diffraction persist even when one single particle passes the two-slit screen at a time. These two features are unique to the quantum mechanics behavior of electrons and photons. The examples presented were initially “thought experiments” (used to introduce the quantum mechanics concept), which however have been supported by experiments. They help to gain an understanding of the Heisenberg’s relationship between the the spatial $\Delta x$ and linear momentum $\Delta p$ uncertainties.

In a given experiment, there is also a relationship between the uncertainty $\Delta E$ of the particles’ energy, and the time $\Delta t$ needed to measure the particle’s energy. This Hesenberg’s uncertainty is illustrated through the analysis of a refraction grating. In order to discern the energy content with a given specified precision, a corresponding minimum measurement time is needed.

5.1 Failure of classical physics to describe atomic phenomena

When attempting to apply classical electrodynamics to atomic phenomena, they lead to predictions that are in conflict with experiment. For example, classical electrodynamics visualizes electrons moving around the nucleus following some classical orbits. But in such a motion the electrons (like in any other case involving accelerated charges) would emit radiation; that is, the electrons would lose their energy continuously, eventually collapsing into the
nucleus. Thus, classical electrodynamics predicts that the atom is unstable, which does not agree with reality.

This contradiction between classical theory and experiment indicates the need of a fundamental modification of the basic physical concepts and laws, so they can be applicable to atomic phenomena (involving small masses, small distances). We have to be prepared, then, that the new theory may at first look peculiar and mysterious to everyone, both to the novice and to the experienced physicist. In consequence, we will learn quantum behavior phenomena in a sort of abstract or imaginative fashion and not by connection with our direct experience with macro-scale objects.\(^1\)

In Chapter 5 we will describe one of those ‘strange’ phenomena, namely, the behavior of electrons passing through a couple of slits, which turns out to be absolutely impossible to explain in classical terms, and which has in it the heart of quantum mechanics.

One striking new feature in quantum mechanics refers to the impossibility in obtaining a perfect knowledge of all the physical variables that relate to the motion of an atomic object (electrons, in the case of the experiment to be described below). There is always an inherent uncertainty in the value of the variables (position, velocity, ...) describing the dynamic of the object. Pictorially, there is an absence of trajectories.

At the beginning, we may get the impression that such lack of determinist description of the electron’s motion has the appearance of a mystery. This mysterious (no deterministic) description of the atomic phenomenon may constitute the only ‘mystery’ in quantum mechanics. Thus, it is worth to consider having an early exposure to it through a “simple” experiment of electrons passing through two slits. A warning must be stated, however; paraphrasing Feynman,

\[ \text{we will not be able to explain how it works,} \]
\[ \text{we will tell you how it works.}^1 \]

That is to say, we will provide (this will actually be done in Chapter 6) a mathematical procedure (based on the concept of amplitude probabilities) that when applied to a given problem (an electron passing through a couple of slits, for example) it predicts an outcome that happens to coincide with the experimental results (this will be done in terms of “amplitude probability” concept), but we may still leave you wondering “what exactly the electron does”? But, before undertaking a description of the amplitude probability concept (to be discussed in Chapter 6), in Chapter-5 we highlight first the shortcomings of
classical tools for describing the behavior of electrons, interpret the experimental results based on the Heisenberg’s principle, and close up with a summary contrasting the different classical and quantum mechanics approaches to describe the dynamics of a system.

In the following sections, we will contrast the quantum behavior of atomic particles (electrons) with the behavior of macroscopic particles (gun bullets), and make this difference in the context of the more familiar behavior of waves (using light interference as a specific example.)

5.1.A The concept of Probability. An experiment with bullets

A gun shoots a stream of bullets, randomly and over a large angular spread. Between the gun and the screen there exists a wall with two slits big enough to let one bullet to pass through. A detector at the screen serves to count the number of bullets arriving at each position \( x \).

Some bullets may go directly straight through any of the holes, or bounce on the edges, etc. Thus there is some uncertainty about the exact position “\( x \)” at which a particular bullet will arrive at the screen.

One way to deal with uncertain quantities is to use the concept of probability. Let’s introduce the concept of probability through the analysis of different possible scenarios that can occur when using the experimental setup presented above.
• Let’s block the slit #1. What would be the probability \( P_2(x) \) that a bullet passing through the slit #2 arrives at the position \( x \)?

The answer requires to make an experimental procedure:

If out of a total number \( N_{\text{total}} \) of incident bullets, the number of bullets hitting the screen at \( x \) is \( N_2(x) \), then the probability \( P_2(x) \) is defined as the ratio \( P_2(x) \equiv \frac{N_2(x)}{N_{\text{total}}} \). But for the concept of probability to acquire a useful meaning the total trial number \( N_{\text{total}} \) has to be very large. Symbolically,

\[
P_2(x) \equiv \lim_{N_{\text{total}} \to \infty} \frac{N_2(x)}{N_{\text{total}}} \quad \text{Definition of probability} \tag{1}
\]

Fig 5.1 above shows a schematic of the expected behavior of \( P_2(x) \).

• Let’s block slit #2. What would be the probability \( P_1(x) \) that a bullet passing through the slit #1 arrives at the position \( x \)?

\[
P_1(x) = \lim_{N_{\text{total}} \to \infty} \frac{N_1(x)}{N_{\text{total}}} \tag{2}
\]

• When the two holes are open, what would be the probability \( P_{12} \) that a bullet, passing through either of the two holes 1 or 2 on the wall, will arrive at the position \( x \) on the screen?

Classical particles typically behave in such a way that, it turns out, the probabilities are additive; that is,

\[
P_{12}(x) = P_1(x) + P_2(x) \tag{3}
\]

Consequently, we affirm that this phenomenon (that deals with relatively large particles) lacks interference; that is to say,

*the opening or closing of one hole does not affect the individual probabilities associated with one aperture.*

In other words, if a fraction \( P_2(x) = 10\% \) of the particles arrive at \( x \) when the aperture #1 is closed, then when the two apertures are open the fraction of the particles that arrive at \( x \) after passing through aperture 2 is still 10\%.
5.1.B An experiment with light

A source emits monochromatic light waves, which pass first through a couple of slits, and then arrives at a screen. A detector on the screen measures the local intensity $I$ (energy per unit time per unit area) of the incident light. We would like to understand the intensity distribution $I(x)$ established across the screen.

- When one of the slits is covered, a rather simple intensity distribution (either $I_1(x)$ or $I_2(x)$ respectively) is observed, as displayed in the Fig. 5.2 below. But when the two slits are open, a rather distinct wavy pattern $I_{12}(x)$ is observed. Definitely $I_{12}(x) \neq I_1(x) + I_2(x)$

Consequently, we affirm that this phenomenon with light displays interference since the opening or closing of one slit does influence the effect produced by the other slit. In other words,

\[ \text{one slit appear to behave different depending on whether the other slit is open or closed.} \]

![Light source](image1)

![Wall](image2)

![Observation screen](image3)

**Fig. 5.2** An experiment similar to the one depicted in Fig. 5.1, but with light.

- **Wave interpretation.** We could attempt to explain this result by considering the interference between the light beams that pass through the slit 1 and the slit 2. If the beams arrive at the observation screen in phase they will produce a bright spot; if they arrive out of phase a dark spot will result.
Notice, this plausible interpretation needs the simultaneous participation of two entities (two beams).

- What about if, in addition, we take into consideration the constituent granularity of light? How does this new information affect our conclusion made in the paragraph above?
  In such a case, we can still argue that photons passing through the two different apertures would interfere at the screen.
  However, if we deem the intensity of the light source such that only one photon is emitted at a time (and employ a large exposure time so that we still collect a lot of them at the detection screen), what would be the result?
  Based on the plausible wave interpretation stated above, the prediction is that the wavy interference pattern $I_{12}(x)$ would not form since “a single photon passing through a slit would have no other photon to interact with.” As it turns out, this prediction is wrong.

- This is what happens when only one photon at a time is emitted from the light source:
  - If a photographic plate is placed on the screen, and the exposure time is chosen as short as to receive only one photon at a time, it is observed that:
    
    \[
    \text{Each photon produces a localized impact on the screen. (4)}
    \]
    
    That is, a single photon does not produce on the screen a weak interference pattern like $I_{12}(x)$ (that is, a single photon does not spatially spread out); it produces just a single localized spot.

  - When the exposure time is drastically increased as to capture a large number of photon, a progressive formation of a wavy pattern $I_{12}(x)$ is observed!
    This was demonstrated (although in an experimental set up different than the one being described here) by G. I. Taylor in 1909, who photographed the diffraction pattern formed by the shadow of a needle, using a very weak source such that the exposure lasted for months.
It can be concluded then that interference does not occur between photons, but it is a property of single photons. The photons arrive to the photographic plate in a statistical fashion and progressively build up a continuous-looking interference pattern.

This has been confirmed in more recent experiments performed in 1989 by A. Aspect, P. Grangier, and G. Roger.

- **Why the pattern is affected so drastically when the two slits are open?**
  I speculate (we will improve our speculation later on in Chapter-6 when the concept of amplitude probability is introduced):
  A somewhat spatially-spread wavefunction $\Psi_{\text{ensemble}}^{\text{IN}}$ is associated to an ensemble of photons of wavelength $\lambda$ incident on the two apertures. ($\Psi_{\text{ensemble}}^{\text{IN}}$ depends on the aperture locations, separation between the apertures, etc)

  As a photon approaches the screen, its behavior is governed by the wavefunction $\Psi_{\text{ensemble}}^{\text{IN}}$ (spread over the space on the left side of the apertures) and a wavefunction $\Psi_{\text{ensemble}}^{\text{THROUGH}}$ spread at the right of the apertures.) We picture then a wavefunction $\Psi_{\text{ensemble}}$ passing not only through one slit but through the two of them. That is, the wavefunction $\Psi_{\text{ensemble}}$ (composed of $\Psi_{\text{ensemble}}^{\text{IN}}$ and $\Psi_{\text{ensemble}}^{\text{THROUGH}}$) is intersected by the two apertures. (The spatial localization of $\Psi_{\text{ensemble}}$ could be assumed to be of the order of $\lambda$. Consequently, a slit-separation of the order of $\lambda$ would lead to more significant effects on $\Psi_{\text{ensemble}}$.)

  Accordingly, the wavefunction $\Psi_{\text{ensemble}}^{\text{THROUGH}}$ at the other side of the screen carries the information about the two slits. In a way, we can say that the initial wavefunction $\Psi_{\text{ensemble}}^{\text{IN}}$ was modified and became $\Psi_{\text{ensemble}}^{\text{THROUGH}}$. The latter has the fingerprints of the two-slit. But let’s keep in mind that $\Psi_{\text{ensemble}}$ (composed of $\Psi_{\text{ensemble}}^{\text{IN}}$ and $\Psi_{\text{ensemble}}^{\text{THROUGH}}$) describes the photon’s behavior but in a statistical manner. Another way to say this is that $\Psi_{\text{ensemble}}$ has meaning only to describe an assemble of incident photons). That is, sometimes the incident photon is modified in one way and, accordingly, it
statistically arrives at some place $x$ on the screen (the statistics is governed by $\Psi_{\text{ensemble THROUGH}}$). Another incident photon is modified in potentially different way and arrives somewhere else $x'$. Overtime, the collection of many photons builds up the pattern $I_{12}(x)$.

**5.1.C An experiment with electrons**

Let’s consider an experiment in which mono-energetic electrons strike a wall having two slits; beyond the wall a screen full of detectors (only one is shown in the figure) allows recording the spatial pattern distribution of electrons-arrival along the screen.

The arrival of an electron at the detectors is sensed as a detector’s ‘click’.

Electrons arrive at the detector in identical pieces (one ‘click’ per electron).

When one electron at a time is emitted by the source, only one detector’s ‘click’ is detected at a time.
The results sketched in the diagram above indicate that,

\[ \text{electrons behave like light.} \] (6)

Indeed interference pattern due to incoming single electrons was observed in 1989 by A. Tonomura, J. Endo, T. Masuda, T. Kawasaki, and H. Ezawa

In summary, the observed interference effect in the detection of electrons and photons (as described above) indicate that the following statement (plausible in the classical world) is not true:

\[ \text{“Each electron either goes through aperture 1 or through the aperture 2”} \] (7)

For:
- When we close aperture-1 (so we know for sure the electron passed by the aperture-2) the detector at \( x=x_0 \) (see figure 5.3 above) collects a given number of electrons, let’s say, \( N_2(x_0) = 348 \).
- When we close aperture-2 (so we know for sure the electron passed by aperture 1) the detector at \( x=x_0 \) collects a given number of electrons \( N_1(x_0) = 1965 \).
- How is that when we leave both apertures open the detector at \( x=x_0 \) collects \textit{none}?

The wavy pattern \( P_{12}(x) \) (as shown in the Fig. 5.3 above) is then not compatible with the statement (7). When the two apertures are
open, we do not know which aperture a specific electron passed through.

This type of uncertainty is rooted in the quantum behavior of atomic objects. Further, it appears that Quantum Mechanics protects this uncertainty as to defeat any attempt to overcome it. In the next section we will describe those attempts in a more formal detail.

5.1.D Attempts to track the trajectory of the electrons’ motion

Watching electrons with a light source.

As shown in Fig. 5.4 below, a light source has been added to the previous set up in an attempt to watch what specific aperture electrons choose to pass through.

**Procedure:** When an electron passes aperture-1, for example, we should observe photons being scattered from a region nearby the aperture 1. These electrons (type 1) will be tabulated under the table-column $P_1’$

Similarly, for electrons scattering photons from a region near the aperture 2 a second tabulated column will be generated.

When this experiment is performed, the result, it turns out, is:

**Result:** $P_{12}’ = P_1’ + P_2’$

Just as we expect for classical particles. (That is, the interference pattern, obtained when we do not watch the electrons, does not occur any more.)

**Conclusion**

“When we watch the electrons, they behave different than when we do not use light to watch them.” (8)
What about using more gentle photons?

Since, as we know, light is an electromagnetic entity (which interact with charged particles), probably we should expect electrons to behave different when using photons to watch them.

- The motion of the delicate electrons may be perturbed by the linear momentum $p = \hbar / \lambda$ carried by the photons.

- An alternative would be to use more gentle photons; those of longer and longer wavelength.

We will encounter a limitation in this approach, however. As we know, when using light there is a restriction on the ability to distinguish how close two spots are apart from each other and still be seen as two separate spots: If the separation between the objects is smaller than the wavelength of the radiation being used, we will not able to distinguish them.

Thus, in our two-slit experiment with electrons,

- as we use longer and longer wavelengths (as to minimize the perturbation on the electron’s motion) our ability to distinguish the slits will be worst and worst. We will not be able, then, to ‘see’ which slit a given electron passed through.

The above observation helps illustrate the unavoidable perturbation role played by a measurement. (We knew this, but we did not expect to be so drastic when working with small particles like electrons.)
5.1.E Discarding the idea that electrons move along paths

Trying to predict the results of the passage of electrons through two slits, one expected, based on classical intuition, that the pattern formed on the detection screen would be a simple superposition of the two patterns obtained when one slit, one at the time, is covered:

*each electron, moving in its path, passes through one of the slits and has no effect on the electrons passing through the other slits.*

The experiment however shows an interference pattern, no corresponding at all to the sum of the pattern produced by each slit separately.

*It is clear that this result can in no way be reconciled with the idea that electrons move in paths.*

Thus, a new description, the quantum mechanics description, must be based on ideas of motion which are fundamentally different from those of classical mechanics. In QM there is not such a concept as the path of a particle. This forms the content of what is called the uncertainty principle, one of the fundamental principles of quantum mechanics, discovered by W. Heisenberg in 1927.

5.2 The Heisenberg’s Uncertainty Principle

It was suggested by Heisenberg that the new law of Nature could only be consistent if there were some basic limitations in our experimental capabilities not previously recognized. Two important cases will be discussed with some detail in the sections below; one pertaining the position and momentum variables, the other Energy and time (indeed, the uncertainty principle involves a pair of variables.)

5.2.A Uncertainty in the position \( \Delta x \) and linear momentum \( \Delta p \).

Originally Heisenberg stated:
If you make the measurement of any object, and you can determine the x-component of its momentum with an uncertainty \( \Delta p \), you cannot at the same time, know its x-position more accurately than \( \Delta x = \frac{h}{\Delta p} \) \hspace{1cm} (10)

Let’s consider a specific example to see the reasons why there is an uncertainty in the position and the momentum if quantum mechanics is right.

**Example:** Particles (photons) passing through a single slit. *We will focus our interest in the vertical component of the particles’ linear momentum, \( p_y \).*

As shown in Fig. 5.5, since the particles (photons) come from a source far away from the slit, they arrive with a linear momentum oriented practically horizontal. Thus,

Before arriving to the slit: \( p_y = 0 \).

However, we do not know with certainty the vertical positions of the particles \( \Delta y = \infty \).

![Diagram of particles passing through a single slit](https://via.placeholder.com/150)

**Fig.5.5** Trying to localize better the particles in the Y-direction, we make them pass through a slit of size “a”.

Let’ use an aperture of size “a” as an attempt to localize better the vertical position of some of the particles contained in the incident beam. That way,
after the particles had passed through the aperture we would know the y-position with a precision equal to \( \Delta y = a \).

However, notice that for the particles passing through the slit we lose information about their vertical momentum (it was \( \Delta y = 0 \)). The reason is that, according to the wave theory, there is a certain probability that the particles will deviate from the straight incoming direction, and rather form a spread diffraction pattern. In other words, there is a probability that the passing particles acquire a vertical momentum.

A representative measurement of the angular spread is given by the angle \( \Delta \theta \) at which the diffraction pattern displays a minimum (see the inclined dashed-line in the figure above). From the diffraction theory, the condition for the occurrence of that minimum is expressed by,\(^2\)

\[ a \Delta \theta = \lambda \]

A corresponding measurement of the spread of the vertical momentum values is then no better than \( p_o \Delta \theta \); i. e.,

\[ \Delta p_y \geq p_o \Delta \theta = p_o \lambda / a \] \hspace{1cm} (12)

Notice, if a smaller value of “a” were chosen (in order to get a better precision of the particles’ vertical position) a corresponding larger spread of \( \Delta p_y \) is obtained.
Introducing the quantum mechanics concept that the momentum $p_o$ is given by $p_o = h/\lambda$, (11) and (12) imply

$$\Delta p_y \Delta y \geq h$$

This expression indicates, the better precision $\Delta y$ in the particle’s vertical position (using a narrower the slit) the greater the spread in the vertical linear momentum (the greater the probability that a passing particle ends up having a very large vertical linear momentum).

**Fig. 5.5 (Continuation)**

**Fig. 5.6** The probability that a particle ends up at $S$ increases as the slits gets narrower.
5.2.B Relationship between the uncertainty of the energy content $\Delta E$ of a pulse and the time $\Delta t$ required for the measurement.

In this section, the working principle of a grating is used to illustrate the energy-time uncertainty principle.

- First, we familiarize with the functioning of a grating, and emphasize its inherent limited spectral resolution (inability to distinguish waves of slightly different frequency,) which in principle has nothing to do with quantum mechanics).

- Then, by associating the radiation frequency (used in the description in the functioning of the grating) with the energy of the radiation, as well as properly interpreting the time needed to make a measurement of the energy content of a pulse with a desired precision $\Delta E$, the energy-time uncertainty principle results.

If you are already familiar with the functioning of a grating, you can skip Section 5.2.B.a and go directly to Section 5.2.B.b. What is important is to learn the spectral resolution power of a grating, and how this limited resolution relates to the energy-time uncertainty principle.

5.2.B.a Gratings

Let’s consider a fully illuminated grating of length $L$ and composed of $N$ lines equally separated by a distance “$d$”
Fig. 5.7 For radiation of a given wavelength, several maxima of intensity \((m=0, 1, 2, \ldots)\) are observed at different angular positions as a result of constructive interference by the \(N\) lines sources from the grating.

Calculation of the maxima of interference by the method of phasors

- **Case: Phasor addition of two waves**

  Wave-1 \(= A \cos(kx) \rightarrow \) phasor-1 \(= \Psi_1 = Ae^{i[kx]}\)

  Wave-2 \(= A \cos(kx+k\delta) \rightarrow \) phasor-2 \(= \Psi_2 = Ae^{i[kx+k\delta]}\)

  \[
  A \cos(kx) = \text{Real}(\Psi_1) \\
  A \cos(kx+k\delta) = \text{Real}(\Psi_2)
  \]

  Notice,

  A path difference of \(\delta\) gives a phase difference of \(k \delta = \frac{2\pi \delta}{\lambda}\)

  \(\lambda\) gives \(k \delta = 2\pi\)

**Fig. 5.8** Real-variable waves are represented by their corresponding complex-variable phasors.

Phasors: \(\Psi = \Psi_1 + \Psi_2 = Ae^{i[kx]} + Ae^{i[kx+k\delta]}\)

Real variable: \(A \cos(kx) + A \cos(kx+k\delta) = \text{Real}(\Psi_1 + \Psi_2)\)
\[ \Psi = \Psi_1 + \Psi_2 + \ldots + \Psi_N = \]
\[ = A e^{i kx} + A e^{i (kx+k \delta)} + A e^{i (kx+2k \delta)} + \ldots + A e^{i (kx+(n-1)k \delta)} \]
\[ = e^{i kx} \left[ A + A e^{i k \delta} + A e^{i 2k \delta} + \ldots + A e^{i (N-1)k \delta} \right] \]
\[ \Psi = e^{i kx} \left[ \sum_{s=0}^{N-1} A e^{i s k \delta} \right] \] (13)

Fig 5.10 shows the graphic interpretation of the result (13).
Condition for constructive interference

This occurs when contiguous interfering waves travel a path difference $\delta$ exactly equal to an integer number of wavelengths

$$\delta = m \lambda \quad (m = 0, 1, 2, 3, \ldots); \quad (14)$$

which take place at the angles $\theta = \theta_m$ that satisfies:

$$\delta \equiv d \sin \theta_m = m \lambda$$

Equivalently, since $k = \frac{2\pi}{\lambda}$, the max of constructive interference occurs when the phase difference $k\delta$ between two contiguous waves is exactly equal to an integer number of $2\pi$;

$$k \delta = m 2\pi \quad \text{Condition for max of constructive interference}$$

where $(m = 0, 1, 2, 3, \ldots)$.

![Diagram](image_url)

**Fig. 5.11** Condition in which the phase difference between two contiguous phasors is $m 2\pi$. They contribute to the max. of order $m$ (see also Fig. 5.12 below).
Condition for evaluating the sharpness of the “zero-th order” peak.

Around a peak of maximum intensity, the intensity drops to zero, thus defining a line thickness. We would like to calculate the angular broadening associated to that thickness of line intensity.

Lets consider first the max of order $m=0$. Using the method of phasors, we realize that,
A path difference gives a phase difference of (between contiguous radiators)

- \( \delta \) \( \rightarrow \) \( k \delta = \frac{2\pi \delta}{\lambda} \)
- If \( \delta = \frac{\lambda}{N} \) \( \rightarrow \) \( k \left( \frac{\lambda}{N} \right) = \frac{2\pi}{N} \)

For \( N \) oscillators \( \rightarrow \) the total accumulated phase would be \( 2\pi \);

But Fig. 5.10 indicates that when the total accumulated phase is \( 2\pi \) the phasor sum would be zero. In other word, an intensity of zero amplitude wave would be obtained if

\[ \delta = \frac{\lambda}{N} \quad \text{Condition for zero intensity} \quad (15) \]

Since \( \delta \equiv d \sin(\theta) \), expression (15) determines the angle \( \theta_{\min} \) that locates the minimum of intensity (next to the maximum of order \( m=0 \)),

\[ d \sin(\theta) = \frac{\lambda}{N} \]

or

\[ N d \sin(\theta_{\min,0}) = \lambda \quad \text{Condition for the angular position (next to the maximum of order } m=0 \text{) for which the light intensity is zero} \quad (16) \]
Condition for evaluating the sharpness of the “$m$-th order” peak.

The angular location $\theta_{\text{min},m}$ of the minimum of intensity closest to the $m$-th order maximum of intensity can be calculated in a similar fashion. Indeed, in the case $m=0$ we chose a path difference of $\delta = \lambda/N$ between two contiguous waves.

For the case $m \neq 0$ let’s choose

$$\delta = m\lambda + \lambda/N$$

(i.e. a path difference a bit bigger than the one that gives the peak of order $m$).

The corresponding phase rotation is

$$k\delta = (2\pi/\lambda) (m\lambda + \lambda/N) = m2\pi + 2\pi/N.$$  

Notice, however, that the latter rotation is equivalent to a net rotation of $2\pi / N$ (since a rotation of $m2\pi$ brings the phasor to the same place.)

$$k \delta \Leftrightarrow 2\pi/N.$$
Thus we are choosing a net phase difference between two contiguous waves equal to \( 2\pi /N \).

Since we have \( N \) phasors, after \( N \) rotations we will have completed a full rotation, which brings the total phasor to zero. This is illustrated in Fig. 14.

**Fig. 5.13** For a given line-space \( d \) and \( \lambda \), the diagrams (top and bottom figures) show the angle \( \theta_{\text{max},m} \) that determines the intensity-maximum of \( m \)-th order (\( d \sin \theta_{\text{max},m} = m \lambda \)). As the angle increases a bit, a minimum of intensity will occur at the angle \( \theta = \theta_{\text{min},m} \), which is determined by the condition \( d \sin(\theta_{\text{min},m}) = m \lambda + \lambda / N \) (see also Fig 5.14 below.).
Fig. 5.14 The diagram shows the condition that determine the angle \( \theta = \theta_{\min; m} \) (closest to the angle \( \theta = \theta_{\max; m} \)) for which the intensity at the far located screen is zero. This occurs when \( d \sin \theta = m \lambda + \lambda/N \).

The condition \( \delta = m \lambda \), or

\[
\delta \equiv d \sin(\theta_{\max, m}) = m \lambda
\]

(17)

gives the angular location of the max of order \( m \).

The condition

\[
\delta' = m \lambda + \lambda/N
\]

(18)

or equivalently,

\[
N \delta' = N d \sin(\theta_{\min, m}) = m N \lambda + \lambda
\]

gives the angular location of the minimum immediately next to the max of order \( m \).

From (17) and (18),

\[
N d \sin(\theta_{\min, m}) - N d \sin(\theta_{\max, m}) = [m N \lambda + \lambda] - [N m \lambda]
\]

\[
N d [\sin(\theta_{\min, m}) - \sin(\theta_{\max, m})] = \lambda
\]

The left side is approximately equal to,

\[
N d [\cos \theta_{\max, m}] \Delta \theta = \lambda
\]
where \( \Delta \theta = (\theta_{\text{max, } m} - \theta_{\text{min, } m}) \equiv \Delta \theta_{\text{width, } m} \) 

\[
\Delta \theta_{\text{width, } m} = \frac{\lambda}{[\text{Nd} \cos \theta_{\text{max, } m}]} \quad \text{half width of the peak of order } m
\]  

**Question:**

Given \( \lambda \), and having found its peak of order \( m \) at the angular location \( \theta_{\text{max, } m} \), whose width is \( \Delta \theta_{\text{width, } m} = \frac{\lambda}{[\text{Nd} \cos \theta_{\text{max, } m}]} \), what would be the value \( \lambda' \) of another incident radiation such that it peaks at \( \theta = \theta_{\text{max, } m} + \Delta \theta_{\text{width, } m} \)?

**Fig.5.15 Illustration of the width of the maximum of order \( m \).**

For \( \lambda' \) we want a maximum of order \( m \) to occur at \( \theta = \theta_{\text{max, } m} + \Delta \theta_{\text{width, } m} \). This requirement is expressed by,

\[
d \sin(\theta_{\text{max, } m} + \Delta \theta_{\text{width, } m}) = m \lambda' \quad \text{(Condition of max of order } m \text{ for } \lambda')
\]

Using a Taylor expansion, \( \sin(\theta + \Delta \theta) = \sin(\theta) + [\cos(\theta)] \Delta \theta \),

\[
d \sin(\theta_{\text{max, } m}) + d \Delta \theta_{\text{width, } m} \cos(\theta_{\text{max, } m}) = m \lambda'
\]

Let's set \( \lambda' = \lambda + \Delta \lambda \)

\[
d \sin(\theta_{\text{max, } m}) + d \Delta \theta_{\text{width, } m} \cos(\theta_{\text{max, } m}) = m(\lambda + \Delta \lambda)
\]

The condition for max of order \( m \) for \( \lambda \) implies \( d \sin(\theta_{\text{max, } m}) = m \lambda \); accordingly,
\[ d \Delta \theta_{\text{width } m} \cos(\theta_{\text{max}, m}) = m \ \Delta \lambda. \]

Replacing the value of \( \Delta \theta_{\text{width } m} \)

\[ d \{ \frac{\lambda}{N \cos \theta_{\text{max}, m}} \} \cos(\theta_{\text{max}, m}) = m \ \Delta \lambda. \]

which gives \( \frac{\lambda}{N} = m \ \Delta \lambda, \) or

\[ \Delta \lambda = \frac{\lambda}{mN} \quad (20) \]

The interpretation of this result is:

if \( \lambda \) is increased to \( \lambda' \) (by an increment of \( \Delta \lambda = \frac{\lambda}{mN} \)),

the angular position of the peak of order \( m \) corresponding to \( \lambda' \)

will lie on

the same angular position of the minimum of intensity closest to

the max of order \( m \) corresponding to \( \lambda \).

Below, we will obtain this result again (in the context of finding

the resolution power of a grating).

### 5.2.B.b Gratings and Spectral Resolution

The figure shows incident radiation of two different

wavelengths. We want to evaluate the ability of a given grating
to distinguish them

![Fig.5.16](image)  

*Fig.5.16* Radiation of different wavelength (or frequency)  
produces maxima of intensity at different angular positions
Each radiation will produce its own independent spectrum (a set of peak of different orders). Let’s focus on a particular order, let’s say the order \( m \).

We know that the angular location of the peak increases with \( \lambda \),
\[
d\sin(\theta_{\text{max},m}) = m\lambda.
\]
Accordingly, if the value of \( \lambda_2 \) is too close the value of \( \lambda_1 \) then their \( m \)-order peaks of intensity (which have some width) may superimpose to the point that it would be hard to distinguish them from each other in the recorded spectrum.

How close are \( \lambda_2 \) and \( \lambda_1 \) are allowed to be and still be able to distinguishing from each other in the grating spectrum?

**Raleigh’s criterion of resolution** offers a standard metric:

The smallest difference \( \Delta \lambda = \lambda_2 - \lambda_1 \) allowed for this two wavelength is the one for which the angular position of minimum of intensity for \( \lambda_1 \) (the minimum closest to its maximum of order \( m \)) occurs at the same angular position of the \( m \)-order \( \lambda_2 \).

The Raleigh’s criterium is illustrated in Fig. 5.16 for the case \( m=1 \).
Fig. 5.17 Raleigh’s criterion of resolution illustrated for the case $m=1$

For the same angle $\theta$, we want to have

i) a maximum of order $m$ for the incident light of wavelength $\lambda_2$

$$d \sin(\theta) = m \lambda_2 \quad \text{or,}$$

$$Nd \sin(\theta) = Nm \lambda_2$$

(21)

ii) a minimum of intensity for the wavelength $\lambda_1$ (a minimum right after its max of order $m$)

$$N d \sin(\theta) = mN \lambda_1 + \lambda_1$$

(22)

(21) and (22) imply, $Nm \lambda_2 = Nm \lambda_1 + \lambda_1$. Hence, $Nm(\lambda_2 - \lambda_1) = \lambda_1$

Using $\Delta \lambda = \lambda_2 - \lambda_1$ the Raleigh’s criterion implies,

Light containing radiation of wavelength $\lambda$ is incident on a grating of $N$ lines, producing a spectral line of order $m$.

If the incident light contains radiation of wavelength $\lambda'$ that differs from $\lambda$ by a quantity $\Delta \lambda$, it will be distinguished (according to the Raleigh criteria) if $\Delta \lambda$ satisfies,

$$\Delta \lambda \geq \frac{\lambda}{Nm}$$

(23)
Equivalently, to distinguish the presence of two waves whose wavelength difference is $\Delta \lambda$, requires a grating of the following number of lines

$$N \geq \frac{1}{m} \frac{\lambda}{\Delta \lambda} \quad \text{(when using the maxima of $m$-th order)} \quad (24)$$

Since $\frac{\Delta \lambda}{\lambda} = -\frac{\Delta \nu}{\nu}$, this expression can be written also as

$$\Delta \nu \geq \frac{\nu}{mN} \equiv (\Delta \nu)_{\text{min}} \quad (25)$$

This expression gives the minimum frequency-difference between two waves that a grating of $N$ lines is able to distinguish as, in fact, two distinct incident frequencies (when working around the maximum of order $m$).

5.2.B.c Condition for the minimum length time $\Delta t$ required for the measurement of the energy $E$ with a resolution $\Delta E$. 

- Consider a grating of $N$ lines (separated by a distance $d$ from each other).
  
  Let $\lambda = \lambda_1$ (or equivalently a given frequency $\nu = \nu_1$) for which the grating produces a maximum of intensity of order $m$ (which occurs at an angular orientation $\theta_{\text{max}, m}$ given by $d \sin(\theta_{\text{max}, m}) = m \lambda_1$).

Notice in the graph below that in order to fully exploit the coherent interference from the $N$ oscillators we have to require that the spatial extent of the incident wave-train exceed a given minimum length $l = AQ = mN\lambda$. 

Fig. 5.18 Given a grating composed of N lines, for the N oscillators to participate in the m-th order coherent interference the spatial extent of the incident wave-train radiation has to exceed a distance \( l = mN \lambda_1 \). (Otherwise, a shorter wave-train will exploit only a fraction of the N lines in the grating).

- Now, we notice something mathematically curious. The time \( T_{AQ} \) needed by the light to travel the distance AQ is,

\[
T_{AQ} = \frac{AQ}{c} = \frac{Nm \lambda}{c} = \frac{Nm}{\nu}
\]

Using the result in (25), the expression above for \( T_{AQ} \) can also be written in terms of \( \Delta \nu_{\text{min}} \) (the smallest frequency difference the grating can resolve),

\[
T_{AQ} = \frac{1}{\nu / (Nm)} = \frac{1}{\Delta \nu_{\text{min}}}
\]  

- This mathematically curious result, lead us to an important experimental interpretation, as illustrated in Fig. 5.19. In order to resolve the frequency content of the incident radiation with a resolution \( \Delta \nu \), the measurement has to last at least a time \( T_{AQ} = 1/\Delta \nu \), or greater. That is,
\[(\Delta t)_{\text{measurement}} \geq \frac{1}{\Delta v}\]

(Otherwise, if the incident radiation lasted shorter than \(L = mN\lambda_1\), then not all the N lines would participate and the effective resolution of the grating would be less).

Fig. 5.19 Aiming to distinguish the two frequencies \(\nu_1\) and \(\nu_2\) (\(\Delta\nu = \nu_2 - \nu_1\)) a setup to detect the maximum of order m, is chosen. Such constructive interference requires the simultaneous participation of the N wave components in the interfering wavefront CQ. As illustrated also by Fig. 5.18 above, the formation of such wavefront CQ requires a pulse duration \(\Delta t\) of greater than, or at least equal to, the time required by the light to travel the distance AQ, which is \(\Delta t = AQ/c\). It turns out \(AQ/c = 1/\Delta\nu\).

We have arrived to a very interesting interpretation:

\textit{In order to find out whether an incident radiation contains harmonic wave components whose frequencies differ by } \Delta\nu, \textit{the minimum time duration of the pulse has to be } 1/\Delta\nu. \tag{26}

The less uncertainty we want to have concerning the frequency components in the pulse, the longer pulse we need (i.e. a longer measurement-time will be required.)
We would need more (measurement) time if we want to find out the spectral content with more precision (i.e. with less uncertainty).

\[ T_{\text{min}} = \frac{1}{\Delta \nu} \]

**Fig. 5.20** There is a close relationship between the precision \( \Delta \nu \) with which we want to know the spectral content of a pulse, and the minimum time duration \( T \) that the pulse is required to have in order a measurement with such spectral precision: \( T_{\text{min}} = \frac{1}{\Delta \nu} \).

This is also illustrated in the Fig. 5.19 where it is assumed to know the frequency \( \nu_1 \) is contained in the pulse \( f \) (because it produces a big max of interference.)

**Fig. 5.21** The shorter the pulse \((\Delta t)_f\), the larger the uncertainty in knowing the frequency content of the pulse. This occurs because of the short time \((\Delta t)_f\) available for the measurement.

Notice, the results just described have nothing to do with quantum mechanics. It is a property of any wave. It occurs, however, that quantum mechanics associate a wave character to the particles; hence the variables associated to the
wave-particle (momentum, position, energy) become subjected to the frequency/time of position/momentum uncertainties. For example, applying the concept of quantum mechanics $E = h\nu$, expression (25) can be expressed as,

$$(\Delta E)(\Delta t)_{\text{measurement}} \geq h$$

(27)

5.2.C. More examples. Uncertainty principle and the resolving power of a microscope.

5.2.C.a Image formation and the resolving power of a lens

- **P** focuses at T
  A condition for a clear focused image formation is that rays take an equal time to travel from the source point to the image point.

- A point **P’** will focus at a region very close to T.
  If **P’** is too close to **P**, their image will superimpose and become too blur to distinguish them; they would appear to be the same point.
  How close is too close?

![Fig. 5.22](image)

*Fig. 5.22 A point P is imaged by the lens on point T. How far apart have to be a point P’ so that the lens clearly image it at a point different than T?*

Let’s call $t(PST)$ the traveling time of light to go from **P** to **T** passing through **S**.

- The condition that point **P’** focuses at a different point than **T** is that
$t(P'ST)$ and $t(PST)$ have to be different.
(Otherwise they would focus at the same point.)

Again, how much different do $t(P'ST)$ and $t(PST)$ have to be so that we can conclude that these two paths do not belong to a set of rays forming a clear image?

- **The condition of resolution**

  Starting with the path PST and its corresponding time $t(PST)$, let’s calculate the time $t(P'ST)$ as the point $P'$ moves out of axis away from $P$.

  The condition of resolution (the ability to distinguish $P$ and $P'$) states that

  “Two different point sources ($P$ and $P'$) can be resolved only if one source ($P$) is imaged at such a point ($T$) that the time for the light rays from the other source ($P'$) to reach that point ($T$) differs by more than one period.” \(^5\) In other words,

  \[
t(P'ST) - t(PST) > \text{one period of oscillation (of the radiation being used for imaging)}
  \]

  \[
t(P'ST) - t(PST) > \frac{1}{v}
  \]

  Using the diagram in Fig. 5.21, and in terms of $d$ and $\theta$, the expression above becomes,

  \[
  \left[ t(P'ST) - t(PST) \right] = d \sin \theta / v = d \sin \theta / (cn),
  \]

  where $n$ is the index of refraction of the medium
Fig. 5.23 Diagram to calculate the difference in travel time by two rays.

Thus, according to (28), the condition of resolution can be expressed as,

\[ d \sin \theta / (c/n) > \frac{1}{\nu} \]

Or,

\[ d > \frac{\lambda}{n \sin \theta} \quad \text{Resolving power of a lens} \quad (29) \]

Thus, \( \lambda / n \sin \theta \) is the minimum distance that two points P and P' need to be separated in order to be imaged as two different points.

The quantity in the denominator is defined as the numeral aperture of the lens,

\[ \text{NA} \equiv n \sin \theta \quad (30) \]

and the resolving power is typically expressed as

\[ \text{Resolving power} \ R = \frac{\lambda}{\text{NA}} \quad (31) \]

5.2.C.b Watching electrons through the microscope

We wish to measure as accurate as possible the position of an electron. We have to take into account, however, that the very act of observing the electron with photons disturbs the electron’s motion.
The moment a photon interacts with an electron, it recoils in a way that cannot be completely determined (there will be an uncertainty in the possible ‘exact’ recoil direction.)

If we detect a signal through the microscope, it would because the photon of wavelength \( \lambda \) (and of linear momentum \( p = h/\lambda \)) has recoiled anywhere within the lens angle of view \( 2\theta \). That is, the \( x \)-component of the photon’s momentum can be any value between \( p \sin \theta \) and \(-p \sin \theta\). \( (\Delta p_x)_{\text{photon}} = 2p \sin \theta \). By conservation of linear momentum, the electron must have the same uncertainty in its \( x \)-component linear momentum \( (\Delta p_x)_{\text{electron}} = (\Delta p_x)_{\text{photon}} \). Thus, the uncertainty in the \( x \)-component of the recoiled electron’s linear momentum is

\[
(\Delta p_x)_{\text{electron}} = 2p \sin \theta = 2(h/\lambda) \sin \theta
\]

But, where is the electron (after the interaction with the photon)? How accurate can we determine its position. At the moment of detection of the recoiled photon the electron must have been somewhere inside the focused region of the lens. But the size of that region, according to expression (29), is not smaller than

\[
\Delta x = \lambda / \sin \theta
\]

The product of the two uncertainties gives,

\[
\Delta p_x \Delta x = 2h
\]

![Diagram of photon and electron interaction](image)

**Fig. 5.24** The numerical aperture \( n \sin \theta \) of the lens determines the uncertainty \( \Delta p_x = 2p \sin \theta \) with which we can know the electron’s momentum.
5.3 Contrasting the (deterministic) classical and (probabilistic) quantum mechanics descriptions

5.3.1 Deterministic character of classical mechanics

In classical mechanics, the state of a system composed by \( N \) point masses is completely described by \( 3N \) Cartesian coordinates \((x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_N, y_N, z_N)\). If the system is subjected to constrains the \( 3N \) Cartesian coordinates are not independent variables. If \( n \) is the least number of variables necessary to specify the most general motion of the system, then the system is said to have \( n \) degrees of freedom.

The configuration of a system with \( n \) degrees of freedom is fully specified by \( n \) generalized position coordinates \( q_1, q_2, \ldots, q_n \). The objective in classical mechanics is to find the trajectories

\[
q_j = q_j(t) \quad j = 1, 2, 3, \ldots, n
\]

Lagrangian approach to find the trajectories

- One expresses first the Lagrangian \( L \) of the system in terms of the generalized position and the generalized velocities coordinates \( q_j \) and \( \dot{q}_j \) \((j=1,2,\ldots, n)\). Typically \( L = T - V \), where \( T \) is the kinetic energy and \( V \) the potential energy.

\[
L(q, \dot{q}, t)
\]

where \( q \) stands collectively for the set \((q_1, q_2, \ldots q_n)\).

- The functional \( S \), defined as

\[
S(q) = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) \, dt
\]

varies depending on the selected path \( q(t) \) that joins the coordinates \( q(t_1) \) and \( q(t_2) \).

- Of all possible paths the system can take from \( q(t_1) \) to \( q(t_2) \), the system takes only one. On what basis is this path chosen?
Answer: The path followed by the system is the one that makes the functional $S$ an extreme (i.e., a maximum or a minimum).

$$\delta S\bigg|_{q=q_c} = 0$$

(38)

Fig. 5.25 Possible motions for a system going from the fixed coordinate $q(t_1)$ to the fixed coordinate $q(t_2)$. The motion $q(t)$ that makes the functional $S$ an extreme (and represented by the solid path in the figure) is the one that the system actually takes.

Applying the condition (38) to (37) leads to a set of $n$ differential equations of second order

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0 \quad \alpha = 1, 2, 3, \ldots, n$$

(39)

These are the equations to be satisfied by the classical trajectory $q_c(t)$ and the latter constitutes the solution we were looking for.

5.3.2 Absence of trajectories in the quantum behavior of particles.\textsuperscript{6}

In the quantum description, in contrast to the classical one, there are not trajectories. It was emphasized at the beginning of this chapter that if assuming electrons followed trajectories it would lead to the collapse of an atom, which certainly it does not occur.
Notice that the experiment illustrated in Fig. 5.6 above reinforces the concept that it is not possible to assign a trajectory of motion (in the classical sense) to the dynamics of a particle. For example, if we attempted to measure the velocity of the incident particles at $x=0$, we would place two small apertures of size “$a$” at two close positions separated by $\Delta$. But after passing the first aperture (which already poses an uncertainty $a$ on the $y$-position at $x=0$), there is an uncertainty about $y$-position of the particle at $x=\Delta$ of the order of $(\lambda/a)\Delta$ or greater. The definition of a $y = y(t)$ trajectory would require choosing a very small value of $a$, but that would worsen the range of possible values for the $y$-coordinate at $x=\Delta$. Hence, there is not possible to establish differences like (final vertical position) – (initial vertical position), because such difference adopts scattered values; the smaller the value of $a$ the more scattered the values of $\Delta y$. (Note: A blurred and imprecise path could be obtained if the coordinates of the particle are measured with a low degree of accuracy. But what we are addressing here is the impossibility to define quantum mechanically, a trajectory of arbitrary precision.)

Hence, in quantum mechanics there is no such a concept as “the velocity of a particle” in the classical sense (i.e. limit of $\Delta y/\Delta t$ as $\Delta t$ tends to zero). The Heisenberg principle (stated in expression (10) above) prevents ascribing a classical trajectory (of arbitrary precision) to a QM particle.

**Fig. 5.26** Limitations to ascribe trajectories to the motion of a quantum particle. The higher precision of the vertical position at $x=0$, the less precise its vertical position at later times, which makes impossible to apply the classical mechanics definition of velocity.
5.3.3 Classical mechanics as a limiting case as well as a requirement for formulating quantum mechanics

Still, as we will see later, in quantum mechanics a reasonable definition of the velocity of a particle at a given instant can be constructed. This is a reflection of the fact that the formulation of the quantum mechanics theory is such that it has the classical mechanics as a limit. Also, it is in principle impossible to formulate the basic concepts of quantum mechanics without any reference to classical mechanics. Thus

“Quantum mechanics occupies a very unusual place among physical theories: it contains classical mechanics as a limiting case, yet at the same time it requires this limiting case for its own formulation”

The fact that a quantum object does not have a definite trajectory means that it has also no other (classically interpretable) dynamic motion characteristics. Therefore, for a system composed only of quantum objects (system of atomic dimensions) it would be impossible to construct a logically independent mechanics theory. It result plausible to conceive then that a quantitative description of the dynamics of a quantum system dynamics has to include the presence of a classical object; that way as a result of the interaction between the quantum system and the classical object, a classical signature will be registered on the latter which could reveal information about the former. The nature and magnitude of the change on the classical object depends on the state of the quantum object, which can serve to characterize the latter quantitatively.

The classical object is usually called apparatus, and its interaction with the quantum object is referred to as measurement. More specifically,

in quantum mechanics a measurement refers to any process of interaction between classical and quantum objects, occurring apart from and independent of any observer.

The accuracy of the apparatus plays a role here; if the accuracy is very poor then the quantum object may behave as a classical object. The experimenter is driven to obtain the maximum accuracy possible. However, the measurement always affects the dynamics of the quantum object, and it is in principle impossible to make its affect arbitrarily small. But this is a logic consequence of the fact the
dynamic characteristic of the quantum object is revealed by the measurement itself; if no modification occur on the quantum object dynamics, then no signature is left on the apparatus. A signal on the apparatus implies a modification on the dynamics of the quantum object.

Thus, whereas in classical mechanics a particle has definite position coordinates and velocity at a given instant, in quantum mechanics the situation is quite different (the better we know the position, the worse we know its velocity, and vice versa). A simultaneous existence of the coordinates and velocity would imply the existence of a definite path, which the QM particles do not have.

In QM the coordinates and velocity of a particle cannot be measured simultaneously exactly. They cannot have simultaneously definite values.

5.3.4 The less detailed quantum mechanics description

As mentioned above, in classical mechanics a complete description of the state of a physical system is obtained by giving all its position coordinates and velocities at a given instant. With these initial conditions, the equations of motion completely determine the behavior of the system at all subsequent times. In quantum mechanics such a description is not possible, since position and velocities cannot be known simultaneously. **The description of a state in QM is determined by a smaller number of quantities than in classical mechanics**, i. e. it is less detailed than a classical description. This has important consequences regarding the nature of the predictions made in QM. Whereas a classical description predicts the future of a mechanical system with complete accuracy, QM gives a less detailed description.

Even is a particle is in a state QM-described in the most complete manner, its behavior at subsequent instances is still in principle uncertain. For a given initial state of an electron, a subsequent measurement can give various results.

The task in QM consists in determining the probability of obtaining some given results upon performing some measurements.

Comment:
It should be kept in mind that in some cases the probability of a given result may be equal to unity, \textit{i.e.} there is a certainty that the result of the measurement will be unique. In fact, all measurement processes in QM may be divided into two classes:\cite{key-8}

- \textit{In one (which contains the majority of cases) the measurements do not lead with certainty to a unique result.}
- \textit{The other class contains measurements where there is an absolute accuracy to obtain a given result (provided that the possible values of the physical quantity are discrete). This occurs when the system is in a stationary state.}

\textbf{5.4 Non-relativistic quantum mechanics description}

So, how does QM describe the dynamic of a system with \( n \) degrees of freedom, which is classically specified by \( n \) generalized position coordinates \( q_1, q_2, \ldots, q_n \) ?

- In QM such a system is specified, at a fixed instant of time, by a wave function
  \[
  \psi(q_1, q_2, \ldots, q_n),
  \]  
  which satisfies
  \[
  \int |\psi(q_1, q_2, \ldots, q_n)|^2 \, dq_1 \, dq_2 \ldots \, dq_n = 1
  \]  
  where the integration extends over all the accessible values of the generalized coordinates \( q_1, q_2, \ldots, q_n \).
  The set of all such wave functions satisfying (41) form a vector space called the Hilbert space \( \mathcal{G} \).

- In addition to depending on the coordinates \( q_1, q_2, \ldots, q_n \) the wave function depends also on the time \( t \), but the dependence on the \( q \)s and on \( t \) are essentially different. The Hilbert space \( \mathcal{G} \) is formed with respect to the spatial coordinates \( q_1, q_2, \ldots, q_n \) only. States of the system at different instants of time \( t, t', t'', \ldots \), are given by different wave functions \( \psi_t(q_1, q_2, \ldots, q_n), \psi_{t'}(q_1, q_2, \ldots, q_n), \psi_{t''}(q_1, q_2, \ldots, q_n), \ldots \), of the Hilbert space. This description that treats the time variable different than the
position coordinates results from the non-relativistic way we are describing quantum mechanics.

- The normalization (41) responds to the probability interpretation of the wave function, as it will be described in more detailed in the coming chapters. Notice two wave functions differing by a phase factor $e^{i\alpha}$ give the same probability (probability to obtain a value in an experimental measurement). Thus, the choice of that phase factor is arbitrary.

- It is a postulate in QM that a complete knowledge of the wave function represents the maximum information about the system. Every property of the system can be derived from its wave function. We should emphasize, however, that even knowing the wave function, the result of measurement is still uncertain; only probabilistic predictions can be made.

In summary

A system, classically described as one of $n$ degrees of freedom, is completely specified quantum mechanically by a normalized wave function $\Psi(q_1, q_2, \ldots, q_n)$ which contains an arbitrary factor of modulus 1. All possible information about the system can be derived from this wave function.

But, how to build the QM wave function? How to use the classical measurements of the physical properties of a system to build the QM wave function?

It turns out the state of a system, specified by a wave function in the Hilbert space, represents a theoretical abstraction. One cannot measure the state directly in any way. What one does do is to measure certain physical quantities such as energies, momenta, etc., which Dirac referred to as observables. From these observations one then has to infer the state of the system.
We will postpone the presentation of a logical procedure to build QM states until we introduce the concept of QM operators (which are quantities corresponding to the classical observables) and their commutation (or not commutation) properties (Chapter 9).
QUESTION:

1. A light pulse of length duration $(\Delta t)_1$ has wave components of energy within the range $(\Delta E)_1$.

   The energy content can be revealed by making the pulse to pass through a spectrometer; let’s call this energy-content (or pulse fingerprint) “Spectrum-1”

A similar pulse passes first through a slit. But the slit is opened only a fraction of time $(\Delta t)_2 = (\Delta t)_1 / 10$. (That is, an experimental attempt is made to reduce the temporal duration of the pulse.)

a) When the resulting pulse passes through a spectrometer, how different the new spectrum-2 would be compared to “Spectrum-1”?

b) Can we say that since the pulse is narrower it has gained components of a wider range of frequencies?

   More bluntly, If a red color light-pulse passes through a slit that opens only an infinitesimal length of time, could we eventually be able to make this pulse to become a white-light pulse?

Answer:

   a) Unless $(\Delta t)_2$ is not is not smaller than the grating’s $1/\Delta v_{\text{min}}$ both spectrum would look the same.

      Making the pulse shorter by narrowing the time the slit is open does not add new frequencies to the pulse.

   b) No, a red-light pulse cannot be made a white-light pulse by this method.

      Making the pulse narrower indeed widens its spectral Fourier components, but this has consequences on the uncertainty to the resolution with which we can determine such spectral response.

      Experimentally, if the pulse is made so narrow that $(\Delta t)_\text{pulse} < \text{grating’s } 1/\Delta v_{\text{min}}$ then the spectral line will be thicker. But this thickening is not because the pulse has more energies (in its spectral bandwidth); rather, it is because the inability of the grating to resolve better a spectral line.

      In short: making the pulse shorter worsen the resolution to know is spectral content upon measuring, that is, increases the uncertainty in $\Delta E$ (which is not the same to say that increases the $\Delta E$ content.)
Grating’s limitation

$\frac{1}{(\Delta v_{\text{min}})}$

The spectrum produced by these two pulses will look alike

The spectral line will start looking much thicker.

---

1 Richard Feynman, “The Feynman Lecture on Physics,” Vol-I, Chapter 37, or Vol III Chapter 1; Addison-Wesley, 1963.

2 In the following section (when dealing with the functioning of a grating) we will justify this condition in more detail


7 L. Landau, page 3.